Testing of Stability and Sensibility of Macroeconomic Models Using MATLAB

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based on his bachelor thesis written
for the economics major at FSV UK
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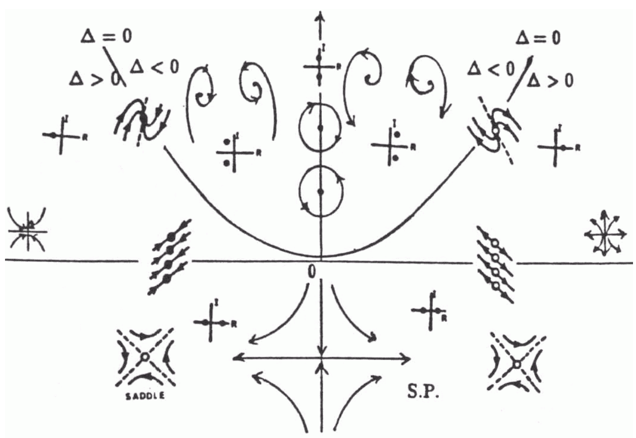

1 Introduction

Macroeconomics is a field of study that analyzes the economy as a whole. In order to describe the economy accurately, very complex models would be required. Nevertheless, even simple models can describe the economy within a required range of accuracy, the simplest of these being linear models, whose solutions are well-known. However, in order to introduce certain features displayed by macroeconomic entities such as cycles and their apparently random length (leading to difficulties with their predictability), non-linear models are required. The price to be paid lies in the fact that these models are more difficult to solve and no general solution type and behavior can be stated beforehand.

One interesting feature of non-linear models is their sensitivity to initial conditions and limited stability; i.e. the solution type of the model depends on the starting point of the economy and may change in response to very small perturbations of the economy such as a single change in the interest rate. These features on non-linearity can be used to model behavior of the limit cases when a variable reaches its maximum or minimum possible value (e.g. capacity of capital goods industries limits investment or the scarcity of resources in general). The importance of non-linear models lies for example in the fact that they can be used to predict stock-market crashes or the length of business cycle, whilst with linear models such predictions are impossible as linear models cannot exhibit such behavior.

The objective of this paper can be divided into several parts. The first task is to formulate a general method of stability testing of any model on the basis of two mathematical theorems. The second task is to apply this test to several classical macroeconomic models and investigate their stability and sensitivity. The investigated models are of the following types:

<u>Macroeconomic Models :</u> Harrod's model and Kalecki's model

<u>Growth models:</u> Solow's neoclassical growth model and Goodwin's growth model

Since the application of the testing method would greatly benefit from the use of a personal computer, a suitable software tool - SEF program in the MATLAB environment - has been developed, which implements the testing method and simplifies the lengthy mathematical operations. Details as to its application as well as its source code for MATLAB are found in Appendix 2.

Last, but not least, the aim of this paper is to provide the author with in-depth knowledge of the above simple models, which is a necessary requisite for the understanding of more advanced models.

The paper is divided as follows. The second chapter describes general features of differential equation models and defines sensitivity and stability. In the third chapter, the method of stability testing is explicitly stated both in mathematical terms and as applied to the SEF program. The fourth chapter contains the focus of the paper – descriptions of macroeconomic models and their stability analysis. The last chapter summarizes the findings.

Advanced reader can start directly with chapter four.

1.1 Notes on sources and notation

Only the sources cited in chapter 6 - Bibliography - were used and all pictures are either author's own work or reproduced from [Tu]. The picture on page 2 is from [Tu], p.138.

Important economic and mathematical terms are in italics, e.g. *structural stability*, definitions are <u>underlined</u>, important results are set in **bold**.

All mathematical variables, expression etc. are printed in italics, e.g. $Y = cY + I + A$.

Dash following a variable's name indicates its time derivative, i.e. $Y'(t) = \frac{dY(t)}{dt}$

Constants are presumed to be real (and in most cases positive), if not stated otherwise.

Abbreviations: DE - differential equation, NL - non-linear (e.g. NL system), L - linear

2 Differential Equations Models and Their Stability

2.1 Mathematical models

Mathematical models are simplifications of reality – they ignore the presence of many entities and bring to the front only a few. Good models strive to sustain the key features and behavior of the modeled situation in order to allow the description of past occurrences as well as the simulation of future possible circumstances and thus they provide the means for the making of predictions. The input (*exogenous*) variables carry input information about the modeled entity (e.g. interest rate, labor efficiency) and the model transforms the information into the output (*endogenous*, determined) variables (e.g. volume of loans, GDP). A good model has as few exogenous variables as possible and at the same time the most endogenous variables. In other words, a good model extracts the maximum from the supplied information.

2.1.1 Static and dynamic models

Static models describe the situation without the explicit use of time. The description may be valid for a single moment in time or throughout a period, the model simply relates one state of input variables to the output ones[1].

On the other hand *dynamic* models deal with time explicitly in their consideration and thus make the introduction of concepts such as 'the next', 'the previous', 'future' and 'past' possible. Consequently one can talk about the history of a system and its description which is most commonly known as *dynamical analysis*[2].

2.1.2 Properties of economic entities

In theory the value of a variable in time could follow almost any path one could imagine. Variables representing economic entities have at least two attributes, which account for the fact, that they try to describe objects originating in the real world.

a) They are *finite* (bounded). This does not imply that a divergent solution (a solution, whose absolute value increases to infinity as time increases) is an unacceptable one[3]. The resolution of this difficulty is not a mathematical one, but rather pragmatic – the description provided by the model is taken to be valid only in a limited interval of time, wherein the solution remains finite. One would argue that it is either necessary to include additional prepositions encompassing scarcity into the model or to replace the model completely when it approaches its real world limits.
b) Economic entities are represented by *real numbers*[4], since they represent real-world objects or their aggregates. In addition they will be confined to economically interesting regions only. For example zero or even negative GDP will be dismissed as opposed to investment, which will be allowed to be negative of course.

[1] For example, if the numbers of employees and machinery utilized are known, one can, given the appropriate model, compute costs of production or total output. However the model does not describe the development of these variables in time and it can be only implicitly inferred that were the number of employees fluctuate in time, the output would follow a similar path.

[2] For example if the model specifies the determinants of the rate of change of total output Y, then the time it takes for output to return to its equilibrium value after a supply shock can be calculated.

[3] On the contrary, most economists would happily see GDP as an always-increasing variable. However the continually sustained growth in its consequences surely leads to a self-contradiction, as the amount of resources is indeed limited.

[4] This does not imply that mathematical constructions, which describe them (e.g. eigenvalues) cannot be complex.

2.1.3 Differential, difference and mixed difference-differential equations

At the beginning of every dynamical analysis, choice has to be made whether to view time as *discrete* or *continuous*. The former entails examining and modeling the situation in successive evenly spaced intervals called periods. Discrete models consist of algebraic expressions, which describe the changes of the observed variable from one period to another (i.e. their differences) as functions of exogenous variables and of the preceding states (e.g. $x_{t+1} = k(x_t - x_{t-1})$). (Hence *difference equations)*. The solution consists of finding an expression consisting of the initially given values only (e.g. $x_t = x_0 . e^{kt}$)

If the second approach is adopted and time is taken as continuous[5], it is necessary to introduce *differentials* instead of the aforementioned differences[6]. The newly created *differential equations - DEs* (e.g. $x'(t)=k.x(t)$) are suitable for modeling of events distributed throughout a period and as well as for evaluation at 'any' time and thus allow to trace the development on every scale[7].

Finally and not surprisingly it is possible to combine the two approaches resulting in a *mixed difference-differential equation* (e.g. $x'(t)=k(x(t)-x(t-2))$). By combining the better of the two approaches these equations are capable of describing economic phenomena with greater ease, combining for example both responses, which take place specific time after a cause or those which respond throughout a period (i.e. *discrete and distributed lags*). The difficulty with difference-differential equations is that they are usually much more difficult to solve.

No general criterion can be given, which of the above model types to use in a particular case. The choice depends on the judgement of the model creator about the prevalent modeled phenomena. For example, if discrete lags are considered inherent in the real-world situation, difference equations are to be used and vice versa – distributed lags imply DEs.

In this paper only models consisting of DEs will be investigated due to the following reasons. First, macroeconomic entities of the real world can be considered as to behave in a continuous manner despite of constituting of 'micro' components, because there are so many of them. The immediate consequence of this premise is that the aggregate step-like functions can be thought of as being continuous[8]. Second, the mathematical theory behind DEs is much more developed and DEs are much easier to work with as opposed to the other approaches. Last, the mathematical theorems used in this paper are most easily expressed for DEs. Adding complicated mathematical theorems would not only bring in little light to the topic, but also add to the volume of this paper without any real contribution.

2.1.4 Solution types

The possible solutions of the types of models discussed later (i.e. models described by DEs) are always continuous[9]. In the case of a single independent variable, they may be subdivided into the following four categories.

Unbounded (divergent). Such a solution can either be *divergent monotonically* (always increasing or always decreasing) or *oscillatorily* (oscillates with ever-increasing amplitude).

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[5] More proper would be 'real' in terms of the number theory, i.e. $t \in (-\infty, +\infty)$.

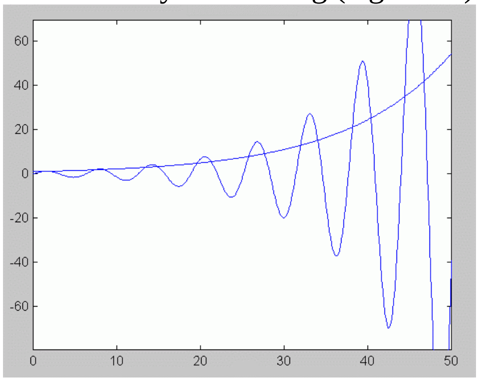
[6] The differences relative to period of unit length are replaced by the limit of differences relative to period, which approaches zero length,i.e. time derivatives are introduced $\lim_{\Delta t \longrightarrow 0} \frac{x(t+\Delta t)-x(t)}{\Delta t} \overset{def.}{=} x'(t)$.

[7] In fact DEs can be thought of as describing the entity not only in fixed points in time $x(t_0)$, $x(t_0+1)$, .. as difference equations do, but also at all times in between $x(t_0)$, $x(t_0+0.314)$, $x(t_0+0.444)$, etc.

[8] For example, in a very small economy, the investment function $I(r)$ could indeed be thought of as step-like, since not many subjects participate. However in a large economy, for every small decrease in interest rate, there exists a subject for whom this decrease renders its investment project attractive. Since the economy is large, the size of this particular project is small in comparison with the aggregate investment and hence the step-like function becomes continuous.

[9]The solutions of DEs are continuous in their range of solution as the existence of the derivative x'(t) implies continuousness of the original function x(t).

Strictly speaking, such a solution cannot be economically acceptable in the long run since it is not bounded. However as one can immediately recognize such a monotonically increasing solution will be intrinsic to simple models describing economic variables only on the basis of the fact that they are observed to be always increasing (e.g. GDP).

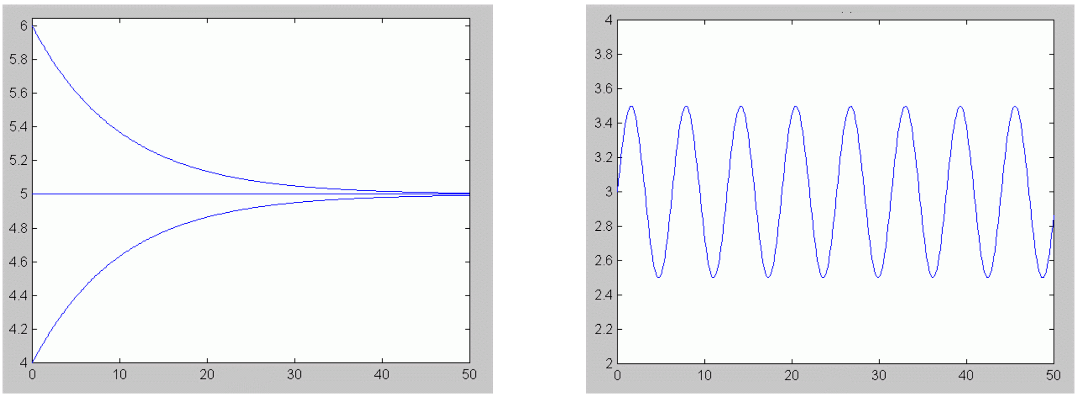


A solution diverging monotonically or oscillatorily.

Finite (Bounded). These solutions express the economic reality better, as they take into account that variables representing real world entities must always be finite numbers. Three separate cases may be observed.

Convergent. Such a solution expresses the tendency of an entity to remain at equilibrium, which is stable, i.e. any external disturbance creates a trend forcing the entity to return to its 'natural' value. For example if no technological progress is assumed, the solution could express the tendency of output or of the rate of growth to return to its natural level (i.e. to potential output, to the balanced growth rate).

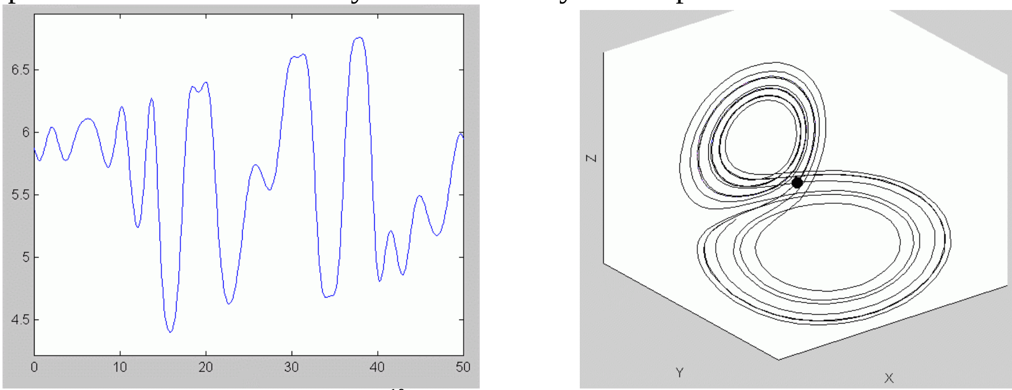
Cyclical. Such a solution is representing an entity, which oscillates around its average value, but never settles to stop at the equilibrium value and overshoots it instead. Business cycles can be modeled either by convergent models, which are repeatedly displaced by external shocks to disequilibrium values (the existence of which may be sometimes dubitable), or precisely by models with cyclical solutions, which thereby show that the oscillatory tendency is intrinsic to the modeled system. A slight disadvantage appears in the fact that these solutions are periodic, i.e. the path in time followed by the variable repeats itself after some time T (its period). Consequently the value of the variable can easily be computed on the basis of knowledge of the behavior of the basic cycle and its period. In other words, the future path of a variable is completely determined by its values in the finite cycle.



A monotonically converging solution (on the left) and a cyclical one (on the right)

Chaotic. Such a solution has no or equivalently infinite period, implying the path in time is always new and unique. It follows that the system responds to all its states in the past and not just the last few. The uniqueness of the time path is coupled with unpredictability in the sense

that the time paths of two systems with close starting points tend to diverge quickly from each other. The unpredictability is a feature, which is believed by most economists, growth models should possess, as the economy also never repeats itself.
In other words, given information about the variable's value in time t, say $I(t)$, the value at later time $I(t+\Delta t)$ can be calculated. However, nothing ensures that for a different, but close starting point $(I+\Delta I)(t)$ the solution at a later time $(I+\Delta I)(t+\Delta t)$ will be close to the previous solution. The uncertainty of economic information implies uncertainty in the location of the starting point and thus makes the dynamics of the system unpredictable.



A chaotic solution in 1-D[10] and 3-D phase portrait (Lorenz's attractor).

2.1.5 Solution types for more complex models

For models with more than one endogenous variable, the aggregate solution is a combination of the separate solutions, in the sense that if plotted on a (n+1)-dimensional graph (n variables and time), the two-dimensional projections with time on one axis and one of the variables on the other would be of the above type.
In order to improve the comprehension of the solution, time axis is often removed from these graphs, which now represent the movement of the variables along their trajectory in the *phase space*. For example the picture above shows the famous Lorenz attractor, which is a graph of the superposition of three bounded chaotic movements in their phase space. The interesting feature of the trajectory is that it is bounded, yet it never repeats itself.

2.2 Structural stability and sensitivity to initial conditions

The questions addressed in this section are of the following type: How will the solution of a macroeconomic model change if one of its parameters changes slightly (e.g. will the economy continue to grow if the interest rate decreases)? Will the economy follow a similar path (e.g. will the growth rate remain the same)? Will the path be dependent on initial conditions (e.g. does the size of autonomous investment or the size of the capital stock matter)? These are very important questions and need to be treated carefully. Before giving a specific answer, the questions need to be formalized.
Suppose that given the values of entities k_1, k_2 determining the investigated variable at time $t = 0$ (for simplicity) $y_0 = y(k_1, k_2, 0)$ the functional relation uniquely determines the solution at all later times $t_1 > 0$ and thus its type in terms of previously defined solution types.
Now suppose that one of the parameters k_1 or k_2 (presume, without the loss of generality, it is x_1) is perturbed a little to $k_1+\Delta k_1$. The question that immediately follows is how will the solution at all later times $t_1 > 0$ look like – will it be similar, i.e. will it be of the same type as the solution for k_1 or will a change of solution type have occurred? If the change to $k_1+\Delta k_1$ does not change the solution type, the system is called locally *structurally stable*. If on the

[10] In fact the graph has been created with the following MATLAB commands: *syms x; ezplot(cos(sin(0.7*x)+cos(x)+0.5*sin(0.3*x-8)-0.05*x)+0.02*x+5,0,50)*

other hand the solution type changes, the system is said to be *unstable*, or equivalently *sensitive to initial conditions* as these may also be treated as parameters of the model.

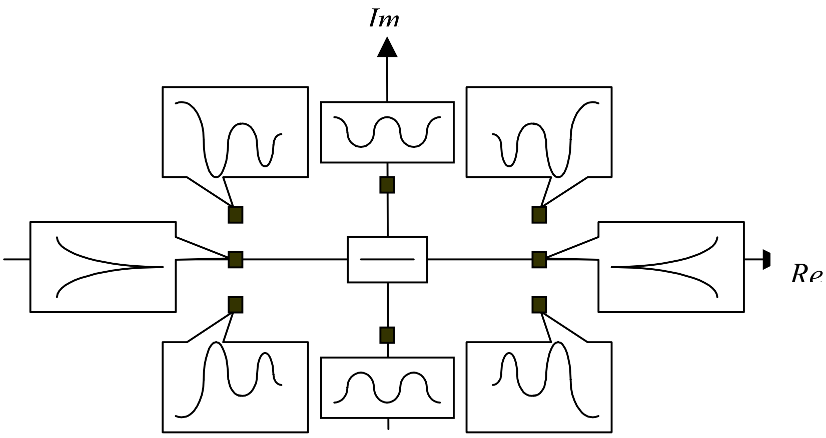
For example the relation $Y = Ae^{kt}$ as a solution to the differential equation $Y' = kY$ could model increasing output and in terms of the previous notation could be written as $Y = Y(A,k,t)$. The constant A is presumed to be positive as zero or negative output have limited economic application. Then the knowledge of Y at $t = 0$ $(Y_0 = A)$ together with knowledge of k, completely determines all future production $Y(t)$ as well as its solution type.

If structural stability is considered, it is clear that if k is fixed then no change of A within its bounds can change the solution type. Consequently it can be stated that the system is locally stable at every 'reasonable' point[11].

Structurally stable systems allow economists to neglect small influences and to work with imprecise numbers as the difference between the predicted value at later time $k(t+\Delta t)$ and the real value at later time $(k+\Delta k)(t+\Delta t)$ is sufficiently small to be neglected.

Continuing with the above example, changes in k can give rise to instability. If k is real, then the solution will be exponentially divergent for $k > 0$, constant for $k = 0$ and exponentially convergent to 0 for $k < 0$. If k is allowed to be complex with non-zero imaginary part (and of course, only the real part of the solution is considered), oscillations will arise. Again these will be exponentially divergent for $Re(k) > 0$, cyclical with constant amplitude for $Re(k) = 0$ and exponentially convergent to 0 (i.e. damped) for $Re(k) < 0$. Thus the point $k = 0$ exhibits instability. A small perturbation from its zero value (which corresponds to a constant solution $Y = Y_0$) can dramatically change the solution type of the system, which the following table and the graph in the complex phase space of k summarize.

Solution	*Re(k) < 0*	*Re(k) = 0*	*Re(k) > 0*
Im(k) <> 0	*oscillating, damped*	*oscillating*	*divergent, oscillating*
Im(k) = 0	*convergent*	*constant*	*divergent*



Solution types of Ae^{kt} in the complex space of k.

To return to the originally posed questions, the key question to be answered is whether the solution type of a macroeconomic model changes with small changes in its parameters or not. In other words, this paper will aim to show under what circumstances a model is structurally stable by performing a stability analysis. In order to answer this question thoroughly, two important mathematical theorems need to be recalled.

[11] Strictly speaking for $A = 0$ the solution type changes to a constant, however this has no economic meaning.

3 Stability Testing

The question to be answered before any stability analysis is performed is, why would economists be interested in knowing whether a model is structurally stable or not. Both stable and unstable types of models have certain advantages. Stable (robust) models are better for prediction. Many input parameters are not known precisely, they may be only estimated, and high sensitivity to input conditions would mean that the solution predicted by the model with the estimated parameter would quickly diverge from the real world situation with the parameter's proper value. On the other hand sensitive unstable models appear to be suitable for stock market or foreign currency market modeling, since movements on these markets are often unpredictable and thus cannot be encompassed by linear models.

3.1 Important mathematical theorems

The (Hartman-Grobman) linearization theorem and mainly the Bifurcation Theory theorem form the theoretical foundations of this paper, as they allow to build a simple criterion stability testing. For fuller mathematical treatment refer to Appendix 1.

3.1.1 Linearization

Any system of first-order DEs (generally NL) can by represented by the general equation $\vec{x}' = \vec{f}(\vec{x}, t) : U \subset R^n \to R^n$,

where x is a vector and f a vector function, and may be difficult if not impossible to solve.
Such a system can always be approximated by a linear system, which for time t = t_0 takes the form

$\vec{x}' = A(t_0)\vec{x} + \vec{b}(t_0) : U \subset R^n \to R^n$, or explicitly $x_i' = \sum_{j=1}^{n} a_{ij}(t_0)x_j + b_i(t_0), \quad i = 1 \dots n$

where A is a matrix with constant coefficients a_{ij} and b is a vector. The coefficients a_{ij} can be found by applying Taylor expansion[12] about the point x, which gives $a_{ij} = \frac{\partial f_i}{\partial x_j}(x)$.

Next, a suitable change of coordinates *($x \to y$, $y = x + A^{-1}b$)*, which merely represent a shift of the origin, simplifies the system even further to $y' = Ay$. Systems of this type can be solved quite easily, but a better understanding is gained by applying the following transformation.
A suitable rotation of the axes ($y \to z$, a linear transformation) diagonalizes the matrix and hence the equations transform to

$z' = J'z$ or explicitly $z_i' = \lambda_i z, \quad i = 1 \dots n$

where J is a diagonal matrix with diagonal elements $\lambda_1, .., \lambda_n$ only, which are known as <u>*eigenvalues*</u>. Consequently, the equations are separated and easy to solve.
Thus for any point in the phase space of the system the eigenvalues[13] $\lambda_1, .., \lambda_n$ of the linearization matrix A determine the movement of the system. In other words for any set of exogenous variables, there exist n eigenvalues, which describe the system's dynamics in the n-dimensional phase space of the endogenous variables. Eigenvalues can be complex numbers and if their imaginary parts are non-zero, oscillatory behavior arises. The behavior type for the transformed variables z is summarized by the table below, whose graph can be found in section 2.2.

[12] see [Tu], p. 134-5
[13] which may be functions of exogenous parameters

Solution of $z' = \lambda z$	*$Re(\lambda) < 0$*	*$Re(\lambda) = 0$*	*$Re(\lambda) > 0$*
$Im(\lambda) <> 0$	*oscillating, damped*	*oscillating*	*divergent, oscillating*
$Im(\lambda) = 0$	*convergent*	*constant*	*divergent*

The real behavior of the original system is found by applying the reverse transformations, i.e. rotation of the axes and shift. Appendix 1 contains tables and graphs of solution types for the two-dimensional case.

3.1.2 Regular and critical points

It is easy to see that as long as $f_i = x_i' \neq 0$ the variable x_i will either rise or fall depending on the sign of its derivative f_i and the linear approximation will represent the non-linear function well as it will preserve its sign. Equivalently, the movement of the system in the phase space can be found once its eigenvalues are determined, though in the direction given by the rotated axes. However, for points x^* where the vector function $f(x^*)$ equals zero – these are called <u>*critical*</u> or <u>*fixed points*</u>, as opposed to <u>*regular points*</u> where $f(x) \neq 0$ - the modeling must take a subtler approach than a pure linearization.

Fixed points are points of stability, as the systems tend to remain there. Once a fixed point is reached, nothing induces a change in the variables as all derivatives are zero ($x_i' = 0 \Rightarrow x_i = const_i$). However exogenous parameters can change the form of $f(x^*)$ so that the fixed point suddenly becomes a regular point and the eigenvalues begin to drive the system somewhere else. It is known from mathematical theory[14], that as long as all eigenvalues of the linearization A at the critical point x^* are non-zero (i.e. $det(A) \neq 0$), the linearized and original system respond to the perturbation in qualitatively similar ways, i.e. the resulting movement of the original NL system will correspond to the movement of its linearized system. These critical points, where $det\ A \neq 0$ are called <u>*simple*</u>.

3.1.3 Hartman-Grobman theorem

Simple point with linearization A, whose eigenvalues have no zero real parts (i.e. $Re(\lambda_i) \neq 0$ for all indices i) is called <u>*hyperbolic*</u> and the Hartman Grobman Theorem[15] simply states that the conditions, under which the linearized system is structurally equivalent to the original NL system, can be extended to hyperbolic points.

Its consequence is that near a hyperbolic critical point (as well as near any regular point), the system is *structurally stable* – it is robust to small perturbations, since it behaves as the stable linear system.

3.2 Example and summary

3.2.1 IS-LM example

The dynamic IS-LM model[16] could be represented by the following two DEs:

$$Y' = \alpha(D(Y,i) - Y(Y,i)) \qquad i' = \beta(L(Y,i) - \overline{M}),$$

where Y is total output, i is interest rate, D is aggregate demand, L is liquidity preference and M is a constant money supply. The constants α and β are speeds of adjustment, assumed positive. It follows that the model's linearization A would be:

$$A = \begin{pmatrix} \alpha(D_Y - Y_Y) & \alpha(D_i - Y_i) \\ \beta L_Y & \beta L_i \end{pmatrix} + 0,$$ where lower indices represent partial derivatives.

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[14] See Appendix 1

[15] ibid.

[16] based on [Tu], p. 34, 107-9

In particular if aggregate demand is consumption $C = cY$ plus investment $I = I_0 - ar$ and liquidity preference composes of transactions demand kY and speculative demand $-hr$, where *c, a, k* and *h* are assumed positive, the DEs become:

$Y' = \alpha(I_0 - sY - ar) \qquad i' = \beta(kY - hr - \overline{M})$,

Substituting gives $A = \begin{pmatrix} \alpha(-s) & \alpha(-a) \\ \beta k & -\beta h \end{pmatrix}$ and the equation becomes of the form $y' = Ay$.

Its eigenvalues are[17] $\lambda_{1,2} = 1/2(-h - s \pm \sqrt{(h-s)^2 - 4ak})$. The dynamics of the system depends on its eigenvalues as functions of their parameters *s, a, k* and *h*. The Hartman-Grobman theorem need not be taken into consideration as the system is linear.

3.2.2 Linearization summary

To summarize the above procedure, any system (L or NL) can be approximated at any time by a linearized system $\vec{x}' = A\vec{x} + \vec{b}$, whose 'origin' is found as $-A^{-1}b$. Insight is gained by 'rotating the axes' ($y \rightarrow z$), which separates the equations. Whether the new variables z will converge, diverge from the 'center', cycle around it or remain constant depends on the eigenvalues of the system.

The importance of this linearization is that as long as the real parts of the eigenvalues are non-zero, i.e. $Re(\lambda_i) \neq 0$, the original NL system and the linearized system are structurally equivalent, i.e. the behavior of the general non-linear system can under the above conditions be reduced to that of a linear system with constant coefficients.

3.3 Bifurcation Theory theorem

The only remaining points where linearization is inappropriate are *non-hyperbolic* points. For these points the eigenvalues of the linearized system reside on the imaginary axis.

It is clear that for eigenvalues to the left of the imaginary axis ($Re(\lambda_i) < 0$), the solution will converge and in the opposite case ($Re(\lambda_i) > 0$), the solution will diverge[18]. Consequently if the eigenvalues as functions of a parameter p were to cross the imaginary axis[19] the system would exchange its stability. This phenomenon is called *bifurcation*.

It immediately follows that in the case of non-hyperbolic points linearization is not valid, since the direction of movement of the eigenvalues and the resulting solution type in response to perturbation such as a change in a exogenous parameter are unknown.

The Bifurcation theorem allows to investigate structural stability of a general model in dependence on a chosen parameter $\vec{x}' = \vec{f}(\vec{x}, t, p)$. It states that **if in response to a change of the parameter *p* the eigenvalues cross the imaginary axis at non-zero speed the solution type changes.**

3.4 The method

As has been noted before, as long as the rate of change of a variable is non-zero (i.e. the case of a regular point), the variable continues to increase or decrease depending on the sign of the derivative. For example the investment could be determined by a non-trivial function of several variables $I = K' = f(K, Y, r, \ldots)$, but for every point where investment is non-zero,

[17] See Appendix 1, part 7.2 the trace-determinant-discriminant method.

[18] Disregarding for the moment their imaginary parts, which determine whether the process will be monotonic or oscillatory.

[19] at non-zero speed, i.e. $\frac{d\lambda(p)}{dp} > 0$. This is additional condition required by the Bifurcation theorem for the solution to consist of two pairs of crossing curves.

linearization could be made $K' = \frac{\partial f}{\partial K} K + \frac{\partial f}{\partial Y} Y + \frac{\partial f}{\partial r} r + \ldots$, which would qualitatively determine the future development of capital, i.e. whether it would increase or decrease. The sign of the linearization determines the movement (increase, decrease or zero movement) of the capital stock.

The fixed points are equilibrium points, i.e. once they are reached the movement of the system is zero. However a slight change in exogenous variables can make the fixed point a regular one. Mathematical theorems prove that as long as the real parts of the eigenvalues of the linearization matrix A at the original point x^* are non-zero (i.e. hyperbolic critical point $\mathrm{Re}(\lambda_i) \neq 0$ for all indices *i*), the behavior of the NL system can be reduced to that of a linear system. The Bifurcation theorem then states the conditions for a change of stability to occur for a critical non-hyperbolic point.

The following procedure will be used (the underlined keywords represent the corresponding action in the SEF program):

Analysis steps	
in general	in the SEF program
a) First the model will be converted to the normal form $\vec{x}' = \vec{f}(\vec{x}, t, k_1, k_2, \ldots)$. In Appendix 1 it is shown that any n^{th} order DE can be transformed into a system of n first-order DEs.	
b) Next, the investigated parameter of the model p will be set, i.e. $\vec{x}' = \vec{f}(\vec{x}, t, p)$.	LOAD
c) All critical points will be found, as they are the only points where a change of stability can occur.	
d) Linearization matrix A will be constructed $a_{ij} = \frac{\partial f_i}{\partial x_j}(x, p)$.	
e) Its trace, determinant and discriminant will be calculated.	SHOW t d Delta
f) Its eigenvalues $\lambda_i(p)$ will be calculated.	SHOW l1 l2
g) These eigenvalues will be evaluated at the critical points found in c), so that they remain functions of the parameter p only.	SUBSTITUTE
h) Solution p_0 to the equations $\mathrm{Re}(\lambda_i(p)) = 0$ will be found.	SOLVE
i) The complex graph of $\lambda_i(p)$ will be plotted against the parameter p in the neighborhood of the value p_0, so that the nature of the bifurcation is found.	GRAPH

4 Macroeconomic Models and Stability Analysis

In the following chapter the term *fundamental economic equation* of the model refers to the equation in its simplest form with constants given their economic interpretation, while the term *fundamental mathematical equation* refers to the same equation in normal form with the constants replaced by the notation subsequently used in structural stability analysis using the SEF program in MATLAB.

The investigated models can be sorted either according to their complexity (i.e. linear/non-linear) or according to their dimension (i.e. the number of endogenous variables). Linear models, whose linearization matrix A is identically the same as the original model, are well-known and easy to solve – hence no surprising change of stability is to be expected. The following linear models will be investigated: Harrod's model and Kalecki's model. NL models are only approximated by their linearization and may be expected to be sensitive to initial conditions. The following will be investigated: Solow's neoclassical growth model and Goodwin's growth model.

With respect to dimension, only one- and two dimensional models will be investigated. One-dimensional models are: Harrod's model, Kalecki's model and Solow's neoclassical growth model. The only two-dimensional model analyzed will be Goodwin's growth model. The analysis of this model will be aided by the trace-determinant-discriminant method as described in Appendix 1.

4.1 Homogenous vs. non-homogenous systems

Non-homogenous system can be transformed to a homogenous system by a simple transformation $(x \rightarrow y,\ y = x + A^{-1}b)$ as has been shown in section 3.1.1. These new variables represent *variations* from the point $x = -A^{-1}b$. The transformed system behaves with respect to the origin in the same way the original system behaves with respect to the point $x = -A^{-1}b$. Analysis of the non-homogenous system will be performed only in the simple case of the Harrod model in order to prove the validity of the above reduction.

4.2 Harrod's model

4.2.1 Description

The basic macroeconomic identity upon which the Harrod model as well as the following two macroeconomic models are built is the simple decomposition of total output Y into three parts: consumption C, autonomous investment A and investment I.

In case of Harrod's model the consumption is multiplier-linked: $C = c.Y$, where c is the marginal propensity to consume (MPC), $c = 1 - s$, where s is the marginal propensity to save (MPS) and is usually taken from the interval $(0, 1)$. The investment is accelerator-linked in its most trivial, linear form: $I = v.Y'$. Autonomous expenditure A is assumed to be zero or an exponentially increasing function such as $A_0 e^{gt}$.

There are no time lags, i.e. investment equals savings both *ex post* and *ex ante*.

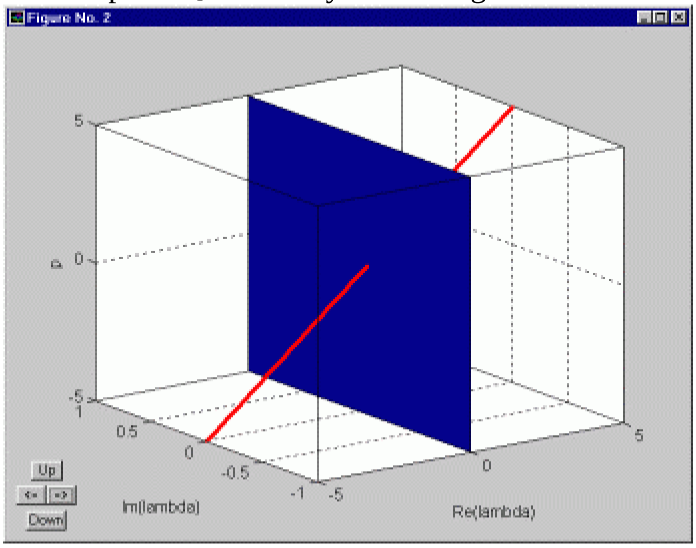
The resulting fundamental economic equation[20] of the model is $Y' = \frac{s}{v} Y - \frac{1}{v} A$.

The fundamental mathematical equation of the model is $Y' = k_1 Y + c(t)$, which is a non-homogenous first order linear DE. The system is one-dimensional (only one eigenvalue) and linear (no change of stability is to be expected). The following analysis may seem too-trivial, yet it is done fully to facilitate the analysis of more complex models.

[20] see [Allen], p. 65

4.2.2 Analysis

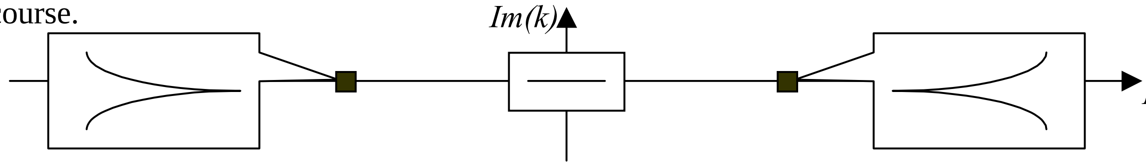
The only parameter of the equation $\dot{Y} = k_1 Y + c(t)$ to be investigated is k_1, thus $Y' = pY + c(t)$, where p is real. The critical points for the homogenous case ($c(t) = 0$) are $k_1 = 0$ and $Y = 0$, the second of which has little economic meaning. As Harrod's model is a linear model, its linearization is the model itself, i.e. $Y' = pY$. As it is a one-dimensional model, the eigenvalue of the one-by-one matrix is the parameter itself $\lambda = p$, which is non-zero for $p \neq 0$ and linearization is thus appropriate in all but single point. Upon substitution of the critical value $k_1 = 0 = p$ into $\lambda(p)$, the solution of it's crossing point with the real axis is found as $p = 0$. Hence for the critical point $k_1 = 0$ the system changes its structural stability.



The eigenvalue λ(p) crosses the imaginary axis for *p = 0*.

The solution to the homogenous equation is $Y = Be^{k_1 t}$, which is structurally equivalent to the output example from section 2.2, from where the solution types can be inferred, with the exponent restriction to real numbers only (and hence no oscillations are possible). B is a constant to be determined from initial conditions such as $Y(0) = Be^0 = B$.

For $k_1 > 0$ the system is divergent/unstable giving rise to an ever-increasing output, for $k_1 < 0$ the opposite is true, i.e. the system converges to zero output. The case $k_1 = 0$ exhibits neutral stability as the output remains constant. The system is structurally stable for all $k_1 \neq 0$, implying a small variation in its value will not change the type of dynamical behavior of the system. However **for $k_1 = 0$ the system is highly sensitive to variations** in this parameter as any small perturbation would set the until-then constant output on a convergent or divergent course.



Solution types of Harrod's model in the phase space of k.

4.2.3 Non-homogenous case

For the non-homogenous case the critical point at a specific time $t = t_0$ is defined by the relation $k_1 Y = -c(t_0) = -c_0 \neq 0$. In this case the reasoning is quite similar once a suitable change of variables is applied. Defining a new variable $Y^* = Y + c_0/k_1$ yields the equation $Y^{*\prime} = Y' = k_1(Y^* - c_0/k_1) + c_0 = k_1 Y^*$, which is homogenous and can be solved by the above method to obtain the same critical point, i.e. $k_1 = 0$. After the reverse transformation is applied, the solution is obtained as $Y = Y^* - c_0/k_1 = Be^{k_1 t} - c_0/k_1 = Be^{k_1 t} + A/(1-c)$, which represents either a converging ($k_1 < 0$) or diverging ($k_1 > 0$) trajectory from the steady state given by autonomous investment only.

4.2.4 Consequences

The economic implications of this analysis are very straight-forward, nevertheless the Harrod model shows the method of stability testing in its simplest form. For values of $k_1 = s/v$ small but positive, the total output is on an ever-increasing course, with the rate of increase equal to s/v. *T*he accelerator contains the explosive element while damping is provided by the multiplier s[21]. The rate s/v is known as Harrod's "warranted" rate of growth as the equation of savings and investment assures such a rate in its outcome.
However, if for some reason MPS were to drop to zero (i.e. no savings would be made and thus no investment would take place), the output until then rising with the rate s/v would stagnate on a constant level in the homogenous case, or it would continue rising with the same rate as autonomous investment would in the non-homogenous case. The accelerator by itself has no effect on the economy.
Moreover, if negative saving/investment would take place implying consumption beyond the current level of output, the output in the following periods would be decreasing to zero level. In another words, for an economy wherein consumption is rising faster than output, which implies decreasing MPS, the rate of growth of output is slowing down and settles at zero if all of the product is consumed and none saved (i.e. zero investment does not extend the economy's actual output). The consumption would stagnate as well ($C = c.Y$). In case the consumption were to continue rising, which in an open economy implies increasingly negative net exports, MPS would turn negative, the solution type would change from increasing (i.e. divergent) to decreasing (i.e. convergent to zero) and total output would fall. **Thus present too-high consumption fuelled by foreign goods in its outcomes leads to lower consumption in the future**.
To summarize the analysis, **in Harrod's model output grows** faster than, with the same rate or slower than the autonomous investment **in dependence on the sign of MPS** (positive, zero, negative).

4.3 Kalecki's model

4.3.1 Description

The economic equations[22] of the model in the specific case of zero time delay[23] $\theta = 0$ are as follows:
Output is determined in the same way as in the previous model $Y = cY + I + A$, however investment is multiplier-linked: $I = a.s.Y - k.K$, where in addition $\frac{dK}{dt} = I$. The first equation defines investment is a fraction of savings, $a \in (0, 1)$, and relates it in the form of a multiplier and not an accelerator as in the Harrod model. This fact could lead to unstable behavior of total output for large values of a[24]. The moderating influence is provided by inverse relation to the size of capital stock expressed by the constant k and although not evident at first, the

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[21] It may appear that the opposite is true, i.e. the greater the accelerator, the slower the divergence and similarly the greater the multiplier, the greater the divergence. However it must be noted that with zero accelerator, the differential equation could not be formed at all and only the multiplier, which is stable or damping in its effect, would influence the economy. See [Allen], p. 68

[22] see [Allen], p. 251-2

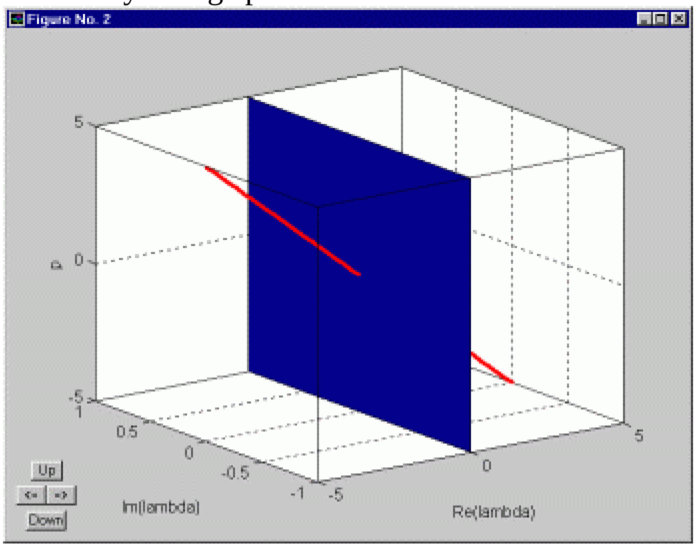
[23] setting $\theta \neq 0$, which expresses a lag between the decision to invest and the actual investment, would lead to a mixed difference-differential equation, which could not be analyzed by methods of this paper.

[24] see [Allen], p. 252

equation has a similar effect as an accelerator would have. The constant k is reasonably assumed to be positive as it expresses capital saturation of the economy.
Upon differentiating of the second equation and suitable substitutions, the following fundamental economic equation is obtained $\frac{dI}{dt} = -\frac{k}{1-a}I + \frac{a}{1-a}A'$.
The corresponding fundamental mathematical equation is formally the same with the Harrod model, i.e. $I' = k_1 I + c(t)$, except that the investigated variable is investment and the corresponding constant k_1 is assumed to be negative.

4.3.2 Analysis

In the homogenous case, i.e. $A' = 0$, which implies zero or constant investment, the stability change point is same as in the previous model: $p = -k/(1-a) = 0$
The non-homogenous case of rising autonomous investment, i.e. $A' > 0$, merely implies the equation is to be interpreted as for differences of investment from an equilibrium value given by $I' = 0$, however the stability change point remains the same.



The eigenvalue λ(p) crosses the imaginary axis for *p = 0*.

4.3.3 Consequences

In the case of constant autonomous investment the solution type for $k \neq 0$ is $I = Be^{k_1 t}$, where k_1 is negative and so the investment tends to converge to zero for any initial value $I(0) = B$.
Given the equations $Y = cY + I + A$ and $I = a.s.Y - k.K$ the existence of equilibrium level of capital and output is implied:

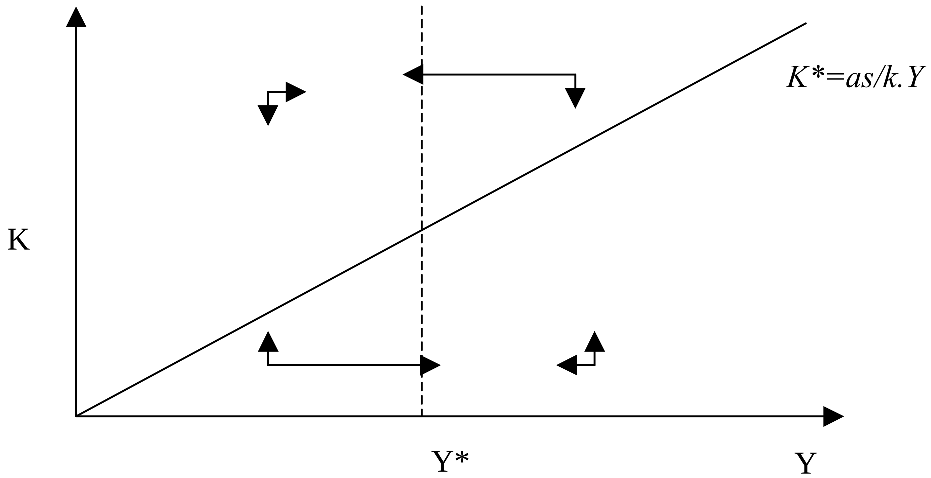
$K = \frac{as}{k}Y$ and $Y^* = \frac{A}{s}$ => $K^* = \frac{a}{k}A$

Five separate initial conditions cases can be distinguished:

a) In case the system starts at the equilibrium output Y* and capital K*, no investment is induced and the dynamics of the system follows autonomous investment.
b) In case output starts above its equilibrium value and the capital stock below the equilibrium value corresponding to this disequilibrium level of output $K = \frac{as}{k}Y$, positive investment will take place ($I = a.s.Y - k.K > 0$), the capital stock will grow, while output will be decreasing despite positive investment.
c) In case output starts above its equilibrium value and the capital stock above the equilibrium value corresponding to this level of output, investment will be negative and the output will decrease fast below its equilibrium level ($\frac{A-I}{s} = Y < Y^* = \frac{A}{s}$).

d) Similarly low output and capital stock above the corresponding equilibrium value imply a capital stock decrease and output increase.
e) Last of all low output and even lower capital stock imply positive investment and the output will increase fast above its equilibrium level.

The following graph for constant autonomous investment shows these movements in the output-capital phase space. The solid line represents the equilibrium level of capital corresponding to a given output $K = \frac{as}{k} Y$, while the dashed line shows the equilibrium level of output, given by autonomous investment.

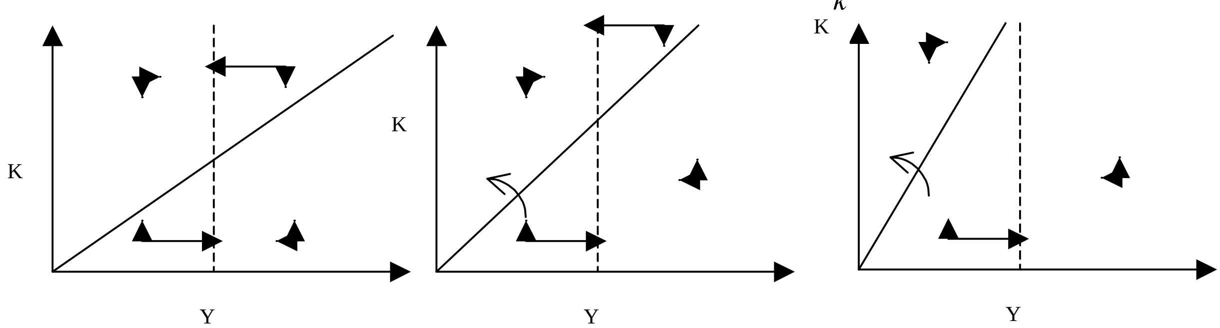


Portrait of the movement of output Y and capital K in the output-capital phase space.

The case of increasing autonomous investment merely implies all the above interpretation is applicable to the differences from a 'background' growth of investment and corresponding output growth.
The point at which structural stability changes, i.e. $k = 0$ implies the following equations: $I = a.s.Y$ and $Y(1-c) = I + A$, which are independent and can be solved to give unique solution: $Y = \frac{A}{s(1-a)}$ and $I = \frac{a}{1-a} A$

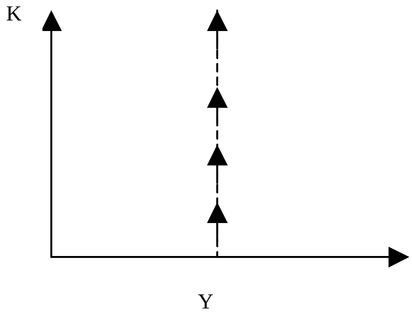
Constant non-zero investment increases capital stock infinitely: $K = \int_0^t I\,dt + K_0 \xrightarrow{t\to\infty} \infty$

The approach of k to zero can be modeled by anti-clockwise rotation of the capital-output line as its gradient approaches infinity for k approaching zero $K = \frac{as}{k} Y$.



Portraits of the movement of output Y and capital K in the output-capital phase space for decreasing capital-sensitivity of investment.

The resulting phase diagram shows that the solution structure changes as only one level of output is now consistent with any capital stock, which increases to infinity.



Portrait of the movement of output Y and capital K in the output-capital phase space for capital-insensitive investment.

To summarize the Kalecki model, **as long as investment depends inversely on capital, the model is stable** and for all initial values of output and capital converges to an equilibrium stable state, which is either constant output, constant capital and zero investment for constant autonomous investment or output and capital rising with the rate of growth of autonomous investment and constant investment. If the capital dependence of investment is broken, **there exists only one possible level of output** and one level of investment consistent with autonomous investment. Capital tends to increase indefinitely.

4.4 Solow's neoclassical growth model

4.4.1 Description

The fundamental economic equation of the model is $r' = s.f(r) - nr$, where r = K/L is the capital-labor ratio, s is the marginal propensity to save, n is the rate of population growth and f is a homogenous output function.

Solow's model may be derived in response to the question whether growth in income is sustainable with full employment of constantly growing population, which is basically answered by finding a suitable capital/output ratio *k* to satisfy the relation $s/n = k$[25].

The model uses the following presumptions/equations:

a) Output function is homogenous of degree one with unlimited substitutability between capital and labor $Y = f(K, L)$ or intensively $Y/L = f(r, 1)$ where *r* is the capital/labor ratio.
b) Saving is a constant proportion of output, $S = sY$, and hence s = both average and marginal propensity to save.
c) Change in capital is equal to investment, $K' = I$.
d) Savings and investment equal ex ante, $S = I$.
e) Labor force is fully employed, $L = L_0.e^{nt}$.

From the last equation and the definition of $r = K/L$ an identity follows $K \equiv rL_0e^{nt}$, which upon differentiation and substitution of the above relations yields the fundamental economic equation, which expresses the fact that the capital/labor ration changes as the net result of actual minus break-even investment $r' = s.f(r) - nr$.

[25] Consider the ratio of marginal (= average) (as in this model S = s.Y) propensity to save s and marginal capital/output ratio k: $\frac{s}{k} = \frac{S}{Y} \Big/ \frac{K'}{Y'} = \frac{Y'}{Y}\frac{S}{K'} = \frac{Y'}{Y}$ as K' = I = S in equilibrium. The task is to equate relative growth in output $\frac{Y'}{Y}$ to the (relative) growth in labor force n (a constant fraction of the population) i.e. n = s/k. More [Gandolfo], p. 198-9

Solow's model can be further enhanced[26] to account for increasing knowledge and capital depreciation to give $r' = s.f(r) - (n + g + \delta)r$, where δ is the rate of capital depreciation and g is the rate of knowledge growth. Although no restrictions are placed upon n, g and δ a priori, their sum is assumed to be positive.
It must be noted that Solow's model is not yet fully analytical and consequently a suitable function has to be used in place of *f(r)* for specific results. A production function satisfying the Inada conditions[27] is for example the well-known Cobb-Douglas production function, i.e. $Y = K^{\alpha}.L^{1-\alpha}$ giving $f(r,1) = f(r) = r^{\alpha}$, where $\alpha \in (0,1)$[28].

4.4.2 Analysis

The fundamental mathematical equation for the Cobb-Douglas case is: $r' = c_1 r^{c_2} - c_3 r$, which can be simplified using the assumption of positivity of $k_3 = n + g + \delta$ to $r' = C(k_1 r^{k_2} - r)$, where $k_1 = s/(n + g + \delta)$, k_2 is assumed from *(0,1)* and C is a positive parameter, which does not alter the solution type only its magnitude and for the purpose of SEF analysis will be thus set to unity.
The critical points are defined by $r' = 0$, which gives the economically uninteresting $r = 0$, and $k_1 r^{k_2 - 1} = 1$, which implies $r = (k_1)^{\frac{1}{1-k_2}}$.

Thus the two critical points are : $r = 0$ and $r = (k_1)^{\frac{1}{1-k_2}}$.

Let k_1 be the parameter. The eigenvalue of the system is $\lambda = pk_2 r^{k_2-1} - 1$, which for the first fixed point $r = 0$ becomes an indefinite expression [29] of the 0^k type. By investigating the limit of $pr^{k_2-1}-1$ for $k_2 \in (0,1)$ and *p* positive, or alternatively from the Inada conditions, it is found that for *r* approaching zero, the eigenvalue of the model increases without bounds. Hence at the origin, the eigenvalue can be thought of as positive and thus the system is repelled from the origin. However if the possibility of zero savings and thus zero investment is accepted (i.e. $p = s = 0$), the limit of the eigenvalue is negative, which implies the opposite conclusion, i.e. the origin attracts the neighboring points.

When the second critical point, which for $p = k_1$ becomes $r = (p)^{\frac{1}{1-k_2}}$, is substituted, the eigenvalue becomes $\lambda = k_2 - 1$. Again the system is stable for the assumed k_2, the eigenvalue is negative, hence the second critical point can be described by the word '*attractor*'[30].
Let k_2 be the parameter. In general the eigenvalue has the value $\lambda = pk_1 r^{p-1} - 1$, which for the first critical point approaches infinity for all $p \in (0,1)$ (i.e. a '*repeller*'), except of the already-mentioned possibility of $k_1 = 0$. The second critical point yields $\lambda = p - 1$, which is in perfect correspondence with the analysis with $k_1 = p$ (i.e. an attractor).

4.4.3 Consequences

The equation $r' = s.r^{\alpha} - nr$ was found to have two critical points. In the first case $r = 0$, the eigenvalue is infinite and the system diverges from the repelling origin with infinite rate. On the other hand, the other critical point given by the condition $s.r^{\alpha} = n.r$, i.e. $r = (k_1)^{\frac{1}{1-k_2}} = \left(\frac{n + g + \delta}{s}\right)^{\frac{1}{\alpha - 1}}$, which implies balanced growth as actual investment equals the

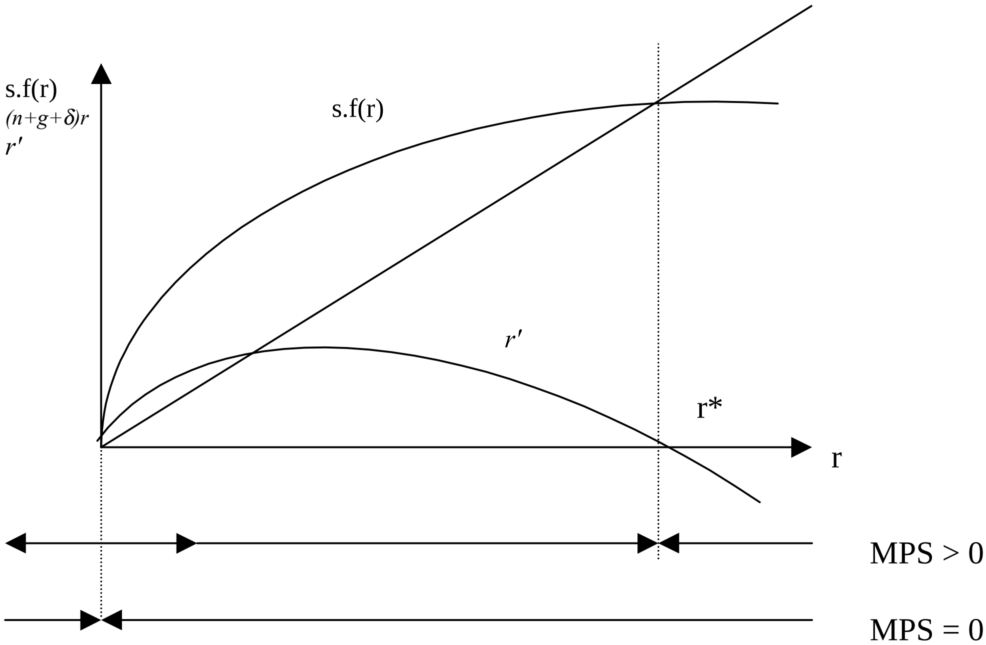
[26] [Romer], p. 7-15
[27] $f'(r) \xrightarrow{r \to 0} \infty$ *and* $f'(r) \xrightarrow{r \to \infty} 0$ More [Inada].
[28] Setting α to zero or unity implies economy without capital or without labor.
[29] In SEF, the expression was found to be NaN, i.e. Not a Number.
[30] In terms of dynamical analysis, critical point with all eigenvalues negative is called an attractor as the solutions tend to converge to this point and similarly, a repeller is a point with all eigenvalues negative.

break-even investment, was found to be an attractor. It follows that the balanced-growth capital-output ratio is a stable solution of the model.

Three separate initial conditions can be distinguished:

a) The capital output ratio equals to the balanced-growth ratio. The system is in equilibrium and tends to remain there.
b) The capital output ratio is greater than the balanced-growth ratio. In this case the system is attracted to the balanced-growth ratio.
c) The capital output ratio is less than the balanced-growth ratio. In this case the system is repelled from the origin and attracted to the balanced-growth ratio.

The necessity of convergence is summarized by the graph below.



Graphs of actual and break-even investment together with two phase space portraits of k for *MPS > 0* and *MPS = 0*.

The above situation was found valid for all economically meaningful values of the two parameters. In addition, it was also found that the system's solution type changes for zero MPS, in which case the two critical points merge at the origin, which becomes an attractor as shown in the lower portion of the graph. This is not very surprising as zero MPS implies zero investment and so gross capital is constant. With the population rising at rate n and current capital depreciating at rate δ, the capital-labor ratio r must slowly decrease to zero. It follows that **as long as there is positive saving** and thus positive investment in the economy, the **capital-labor ratio converges to its balanced-growth value** $r = \left(\frac{n+g+\delta}{s}\right)^{\frac{1}{\alpha-1}}$.

4.5 Goodwin's growth cycle model

4.5.1 Description

Goodwin's growth cycle model is a model of cycles in growth rates and not in absolute levels of national income and as such is an attempt to describe economic growth and cycles more realistically, since in most countries there has been continual growth after the Second World War, though at differing rates. The model is based upon 7 assumptions, the most important of which are:

a) The real wage rate w rises in the neighborhood of full employment.

b) The capital/output ratio $k = K/Y$ is constant, $k \in (0,1)$, which implies $\frac{Y'}{Y} = \frac{K'}{K}$

c) Labor productivity a grows at rate $\alpha > 0$: $Y / L = a = a_0 e^{\alpha t}$, i.e. $\frac{Y'}{Y} - \frac{L'}{L} = \alpha$

d) and labor force grows at rate $\beta > 0$: $N = N_0 e^{\beta t}$, i.e. $\frac{N'}{N} = \beta$

e) All wages are consumed; all profits are automatically invested.

Given c) and d) the employment ratio can be defined as $v = L/N$, $v \in (0,1)$, which implies $\frac{v'}{v} = \frac{L'}{L} - \frac{N'}{N} = \frac{Y'}{Y} - \alpha - \beta$

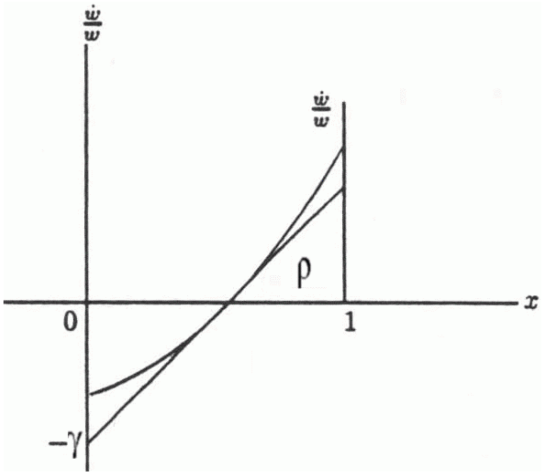
Workers' share of the product u can be defined as $u = \frac{w.L}{Y} = \frac{w}{a}$, $u \in (0,1)$, which implies $\frac{u'}{u} = \frac{w'}{w} - \alpha$.

Wage growth rate, which is a function of employment, can be approximated linearly (see graph on the right) as $\frac{w'}{w} = -\gamma + \rho v$, where γ and ρ are positive constants and thus $\frac{u'}{u} = -\gamma + \rho v - \alpha$

The capitalists' share of the product is *(1-u)* and from e), it follows the rate of profit is given by $\frac{(1-u)Y}{K} = \frac{(1-u)}{k} = \frac{K'}{K} = \frac{Y'}{Y}$

Consequently $\frac{v'}{v} = \frac{Y'}{Y} - \alpha - \beta = \frac{(1-u)}{k} - \alpha - \beta$

The fundamental economic equations[31] of the model are: $v' = \left[\frac{1}{k} - (\alpha + \beta) - \frac{1}{k} u\right] v$ and $u' = \left[-(\alpha + \gamma) + \rho v\right] u$



The wage growth rate function
Picture [Gandolfo], p. 477, Fig. 3.21

Expressed in a simple mathematical form they are known as the Lotka-Voltera equations:

$v' = [c_1 - c_2 u] v$ $\qquad$ $u' = -[c_3 - c_4 v] u$

where in theory c_1 can take on any value as the output/capital ratio can be greater, equal to or less than the sum of labor productivity and labor growth rates, $c_2 > 1$, $c_3 > 0$ and $c_4 > 0$. The equations can thus be rewritten giving the fundamental mathematical form.

$x_1' = C_1 [k_1 - x_2] x_1$ $\qquad$ $x_2' = -C_2 [k_2 - x_1] x_2$

where k_1 can take on any value, $k_2 > 0$ and C_1 and C_2 are positive constants, which for the purpose of SEF analysis will be set to unity.

4.5.2 SEF analysis

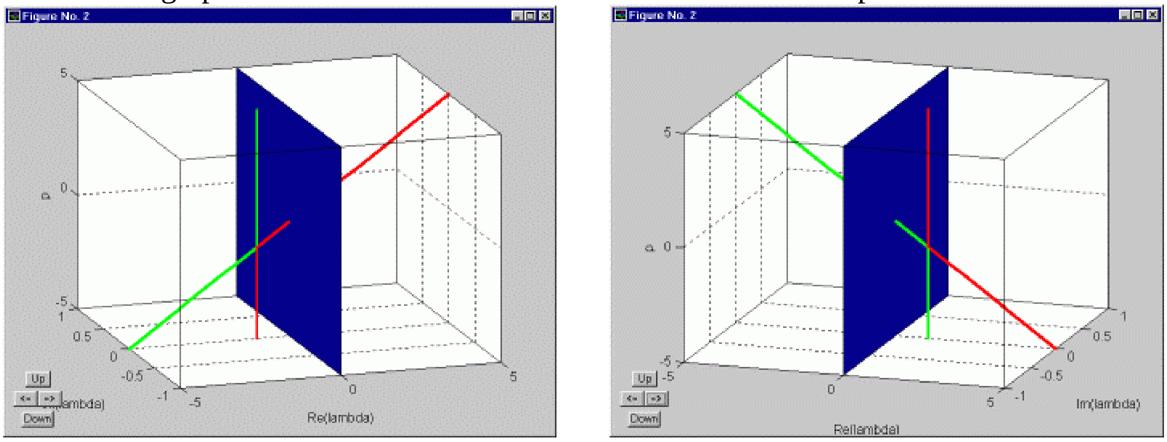
The two critical points are $[x_1, x_2] = [0, 0]$ and $[x_1, x_2] = [k_2, k_1]$.

The first fixed point is economically uninteresting since it represents full unemployment with zero profit for the unemployed workers, nevertheless it will be analyzed to determine the trajectories in its neighborhood. When k_1 is set as the parameter p, i.e. $p = k_1$, the corresponding eigenvalues are $\lambda_1 = p$ and $\lambda_2 = -k2$, as can be observed from the graph below. The system is an unstable saddle point for p positive ($\lambda_2 < 0 < \lambda_1$) and a stable node for p

[31] [Gandolfo], p. 474-481

negative ($\lambda_2 < 0,\ \lambda_1 < 0$). The crossing of the imaginary line takes place at non-zero speed $\frac{d\lambda(p)}{dp} = 1$ and so a bifurcation of the type Saddle-Node takes place.

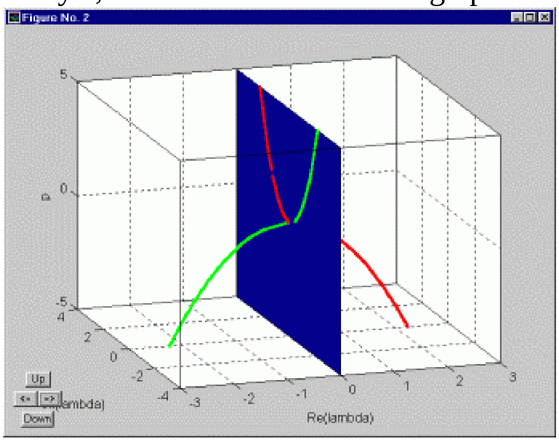
Similarly, when the second parameter is taken to be active, i.e. $\underline{p = k_2}$, the eigenvalues become $\lambda_1 = k_2$ and $\lambda_2 = -p$, as was expected, however as p is always positive, only the top portion of the graph is relevant and no bifurcation is allowed to take place.



The eigenvalue λ(p) crosses the imaginary axis for $p = 0$ for k_1 as the active parameter (on the left), the crossing for k_2 (on the right) is not allowed as $p = k_2 > 0$.

If the second critical point $[x_1,x_2]= [k_2, k_1]$ is considered, with k_1 as the active parameter i.e. $\underline{p = k_1}$, the eigenvalues become $\lambda_1 = \sqrt{-p.k_2}$ and $\lambda_2 = -\sqrt{-p.k_2}$, i.e. for p they are purely imaginary numbers, which implies the point $[k_1, k_2]$ is a center. Strictly speaking for p positive, the linearization is not appropriate as $Re(\lambda_i) = 0$. To confirm that the point $[k_1, k_2]$ is a center and to find whether it attracts, repels or is neutral, further analysis is required.

By setting $k_2 = 1$ for simplicity[32], it is found from the SEF graph that $Re(\lambda_i) = 0$ for all $p > 0$.



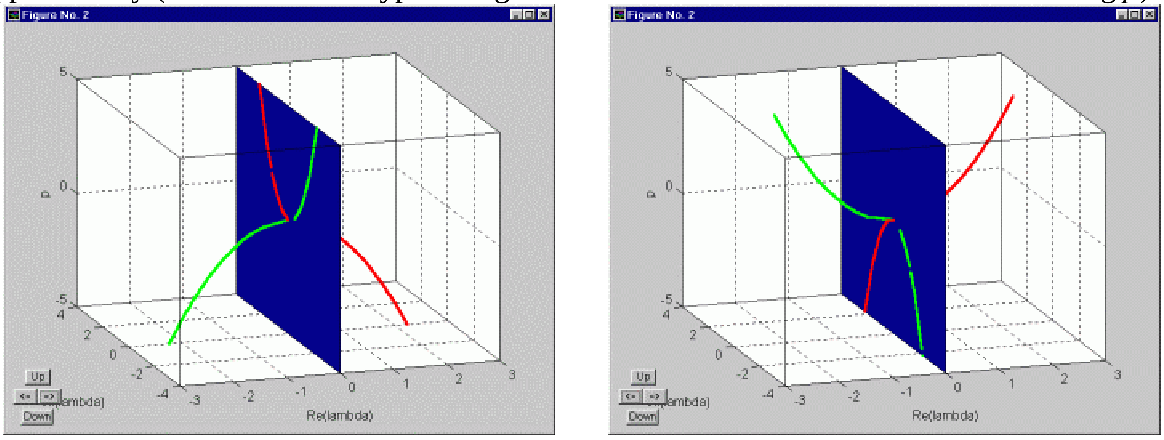
The eigenvalues λ(p) leave the imaginary axis for $p < 0$.

On the other hand for $p < 0$, the eigenvalues are real numbers $\sqrt{-p.k_2} > 0$ and $-\sqrt{-p.k_2} < 0$ and the solution type is a saddle point. However, it cannot be stated at this point that a bifurcation takes place as the crossing speed of eigenvalues is undefined (for p positive the speed is different than that for p negative).

Next, the second parameter will shortly be investigated, i.e. $\underline{p = k_2}$. As the system is anti-symmetrical, it is not surprising to find that $\lambda_1 = \sqrt{-p.k_1}$ and $\lambda_2 = -\sqrt{-p.k_1}$.

[32] The following result is independent of the value of k_2.

Three cases must be distinguished according to the sign of k_1: If k_1 is positive, the trajectory of the eigenvalues of the system is the same as in the previous case. For k_1 equal to zero, the equations break down to $x_1' = -C_1x_2x_1$ and $x_1' = -C_2(k_2 - x_1)x_2$, the eigenvalues are always zero and a different approach must be used. Finally, if k_1 is negative the system behaves in an opposite way (i.e. the solution type changes from Saddle Point to a Center for decreasing p).



$k_1 > 0$: the eigenvalues λ(p) leave the imaginary axis for *$p<0$* *(on the left)*
$k_1 < 0$: the eigenvalues λ(p) come to the imaginary axis for *$p<0$* *(on the right)*

At this point it is necessary to recall that p (= k_2) must be positive and so only the upper portions of the above three graphs are relevant. Thus a solution type change can never occur. The analysis for $p = k_2$ proves the findings of that for $p = k_1$, as for k_1 positive, the system will oscillate around the critical point, for k_1 being zero the system breaks down and for k_1 negative the system is unstable as one of the eigenvalues is positive.

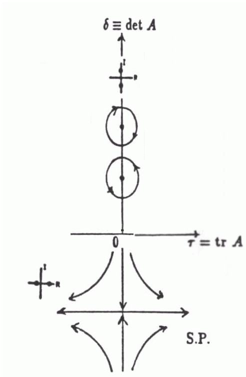
4.5.3 τ-δ-Δ analysis

In order to determine, whether a bifurcation takes place when k_1 changes sign, the system will be analyzed with the trace-determinant-discriminant method (see Appendix 1). As can be noted the description of the method, the solution type changes when τ, δ or Δ change sign.

In this specific case these variables take on the following values:

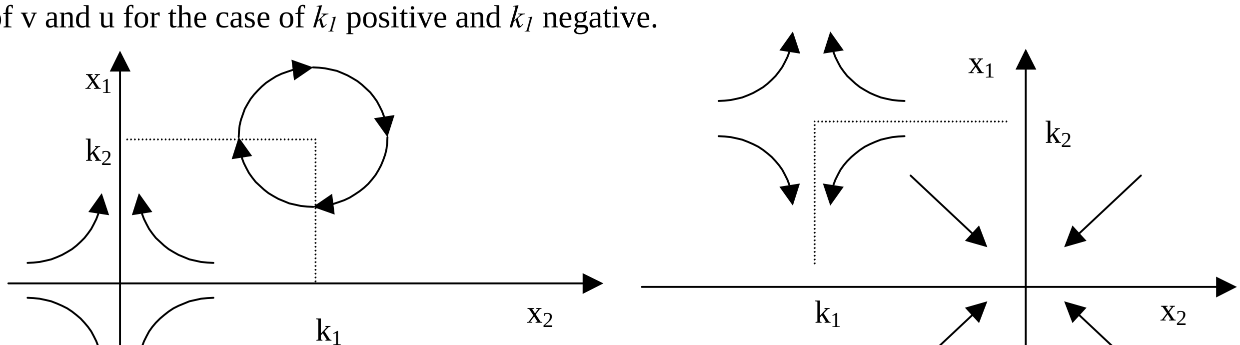
$\tau = k_1 - x_2 - k_2 + x_1$, which is always zero for $[x_1, x_2] = [k_2, k_1]$. Hence, the possible solutions are confined to the δ axis, for all values of the parameters k_1 and k_2.

$7\delta = -k_1k_2 + k_1x_1 + x_2k_2$, which for $[x_1, x_2] = [k_2, k_1]$ reduces to k_1k_2. For $k_1 > 0$, δ is always positive, which corresponds to a stable neutral cycle. On the other hand, for $k_1 > 0$, δ is always negative, which corresponds to a Saddle Point solution as the graph on the right shows.



The δ-axis solutions, Picture [Tu], p.138

The two diagrams below represent the phase diagrams of movements of v and u for the case of k_1 positive and k_1 negative.



Employment rate-workers' share on product phase space trajectory portraits for k_1 positive (on the left) and k_1 negative (on the right).

4.5.4 Consequences

At this point it is necessary to bring back the original equations and give the above analysis economic meaning.

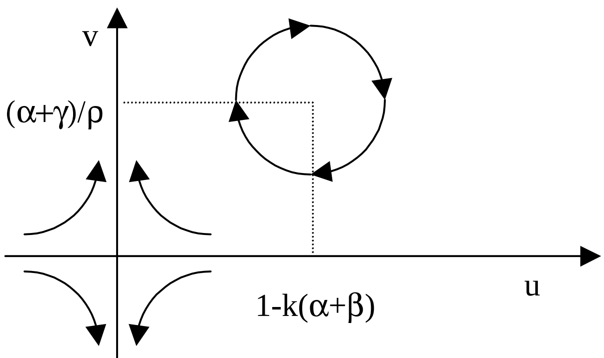
$$v' = \left[\frac{1}{k} - (\alpha + \beta) - \frac{1}{k}u\right]v \qquad \text{specifies rate of change of employment}$$

and $$u' = [-(\alpha + \gamma) + \rho v]u \qquad \text{specifies rate of change of workers' share on product}$$

Since all parameters were assumed to be positive, the only uncertainty was with the sign of the term $k_1 = \frac{1}{k} - (\alpha + \beta)$, which represents the difference between the output-capital ratio > 1 and the effective labor growth rate > 0 (the sum of labor productivity growth α and labor force growth β).

As long as the output-capital ratio is greater than the effective labor growth rate, the eigenvalues are purely imaginary numbers (i.e. a neutral cycle around the point $[k_2, k_1]$) and the system oscillates around its equilibrium value, i.e. the rate of change workers' share and the rate of change of employment oscillate around their average values $[1-k(\alpha+\beta), (\alpha + \gamma)/\rho]$.

As Goodwin pointed out, greatest profits (for the capitalists, workers consume all of their wages according to assumption e)), which are attained when the employment is average, lead to high growth rate. This in turn increases employment to its maximum and it follows that both profits and growth rate diminish (and hence share on the product). Lower growth rate decreases output as well as employment (below its average value), which implies a space for profits to grow is created as productivity is rising faster than the wage rate and hence the employment starts to rise again[33]. This cyclical movement can be observed on the following graph, which shows the trajectories in the workers' profits-employment phase space.



Stable neutral cycle around the point *[1-k(α+β) , (α + γ)/ρ]*
in the employment rate-workers' share on product phase space

It was shown in the above analysis, that if k_1 was to change sign, the system would change its structural stability. For a sudden increase in capital or sudden increase in effective labor (e.g. new technology such as INTERNET), the term $1-k(\alpha+\beta)$ could turn negative[34] (i.e. k_1 could cross the employment axis v). This increase can be summarized by saying that the capital-labor ratio increases[35]. Of course $1-k(\alpha+\beta) < 0$ does not imply that workers' share on profit turns negative as u and v are confined to the interval $(0,1)x(0,1)$.

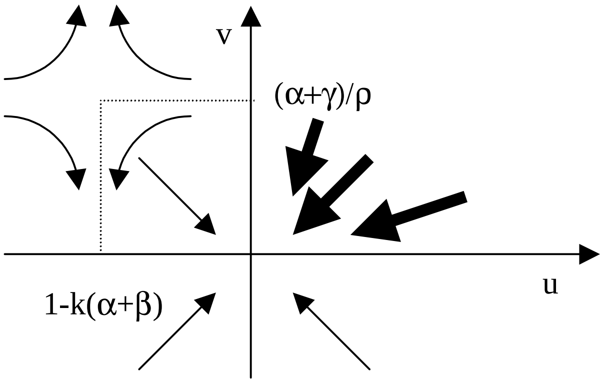
The sudden increase in capital-labor ratio leads to a decrease in the rate of change of employment and increase in capitalists' profits, which implies the workers' share on the product decreases as well. However, as opposed to the above case, when the increases in

[33] [Goodwin] , p. 57-58

[34] Gandolfo presumes the safe upper limits as 5 for k and 0.12 for $(\alpha + \beta)$.[Gandolfo], p. 478 footnote

[35] To prove this is trivial in the first case (K increases), while in the second the increase in effective labor implies less labor is necessary to reach the same level of output (L decreases).

profitability increase employment, the increased profitability only lowers it since the system simply passed a critical value and so the right-hand side of the equation $\frac{v'}{v} = \frac{Y'}{Y} - \alpha - \beta = \frac{(1-u)}{k} - \alpha - \beta$ is negative. The mathematical analysis that such a change is accompanied by the Saddle-Node bifurcation the origin, which becomes a stable attractor and so both rate of change of employment and rate of change of workers' share decrease to the point [u,v] = [0,0], which represents economy where employment is constant as well as worker's wages. The trajectories are shown in the first quadrant of the workers' profits-employment phase space below.



Break-down of the cycle, with the origin as an attractor in the employment rate-workers' share on product phase space

The above depicted collapse of the cycle may appear too horrid, in fact the reality would be different because of two reasons. First, sudden increase in labor productivity is very unlikely as it would imply the ordinary workers are being replaced by sort of 'superworkers'. Even if so fast productivity growth was experienced as to structurally change the solution to the one depicted above, capital can never fully substitute labor[36]. Second, according to assumption b), sudden change in the capital-output ratio is forbidden, even though as Goodwin says, this assumption is of a more empirical, and disputable, sort.

If the capital output ratio is allowed to grow in time *K/Y = k*, i.e. $\frac{K'}{K} - \frac{Y'}{Y} = \frac{k'}{k}$, which implies $\frac{(1-u)Y}{K} = \frac{(1-u)}{k} = \frac{K'}{K} = \frac{Y'}{Y} + \frac{k'}{k}$ the equation determining employment translates to $\frac{v'}{v} = \frac{Y'}{Y} - \alpha - \beta = \frac{(1-u)}{k} - \frac{k'}{k} - \alpha - \beta$ and the final equation would be $v' = \left[\frac{1-k'}{k} - (\alpha+\beta) - \frac{1}{k}u\right]v$, whose instability is even greater as it decreases k_1 (i.e. workers' share on product) even further.

To summarize the analysis of Goodwin's growth model, as long as the output-capital ratio is greater than the effective labor growth rate, both rates of change oscillate around their average values [$1-k(\alpha+\beta),(\alpha + \gamma)/\rho$]. However if the opposite is true, a **Saddle-Node bifurcation takes place at the origin** and the system settles down as both employment and workers' share converge to constant levels.

[36] for if it did, what Goodwin calls Marx's idea of contradictions in capitalism – i.e. the improved profitability already inherently carries in itself its own decrease – would be finally resolved as the ultimate goals of every economic policy would be reached – product without labor. Marx would only have to redistribute capitalists' profits to the workers.

5 Conclusion

This chapter assesses whether the initially stated objectives of the paper were fulfilled and summarizes the stability analysis of the investigated models.

In chapter three a general stability testing method was elaborated and the conditions for structural instability/bifurcation were stated. This general method was subsequently implemented in the SEF program for the MATLAB environment and with its aid several classical macroeconomic models were analyzed in chapter four. Solution types for the analyzed models were stated as well as the critical values of parameters for which a change of stability occurs. Sensitivity to initial conditions was also analyzed.

The solution of the simple **Harrod model** was found to be dependent on the sign of the marginal propensity to save, the critical value of which is $MPS = 0$. For positive values of MPS, the output diverges from the autonomous investment with the rate s/v and for negative values convergence to the autonomous investment with the rate s/v is implied by the model.

The **Kalecki model** was found to be stable as long as investment depends inversely on capital. In this case for all initial values of output and capital the model converges to an equilibrium stable state, which depends on the rate of growth of autonomous investment (i.e. constant output, constant capital and zero investment for constant autonomous investment and constant investment with output and capital rising with the rate of growth of autonomous investment in the case of non-constant autonomous investment). If the capital dependence of investment is broken, i.e. the critical value is $k = 0$, there exists only one acceptable level of output and one level of investment consistent with autonomous investment. Capital tends to increase indefinitely.

The **Solow model** is again dependent on the marginal propensity to save, however it is a very stable model with unique equilibrium that is reached in all but the extreme case of zero savings. Thus the critical value is again $MPS = 0$. However in all other economically meaningful situations with non-zero savings, the capital-labor ratio converges to its balanced-growth equilibrium $r = \left(\frac{n+g+\delta}{s}\right)^{\frac{1}{\alpha-1}}$.

To summarize the analysis of **Goodwin's growth model**, as long as the output-capital ratio is greater than the effective labor growth rate, both the rate of change of employment and the rate of change of workers' share on the product oscillate around their average values [1 - $(\alpha+\beta),(\alpha + \gamma)/\rho$]. However if the opposite is true, a **Saddle-Node bifurcation takes place at the origin** for the critical value $1/k = (\alpha+\beta)$, i.e. output/capital ratio equals the effective labor growth rate, and the system settles down as both employment and workers' share converge to constant levels. This result is not that sensational as may seem at first as today's economies are far from reaching the critical value.

Most importantly it was shown that the stability testing method employed in this paper can be applied to any mathematical model, economical or other. The SEF program can be used to analyze any mathematical model including more advanced economic one- and two-dimensional models. Should higher dimension models be analyzed in the future, the conversion of the program is not difficult. This opens a great space for future economic analysis of other, more advanced macroeconomic models, which display richer, more interesting and in their consequences more precise results and descriptions of the economy.

6 Bibliography

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Appendix

7 Appendix 1 – Mathematical Formulations

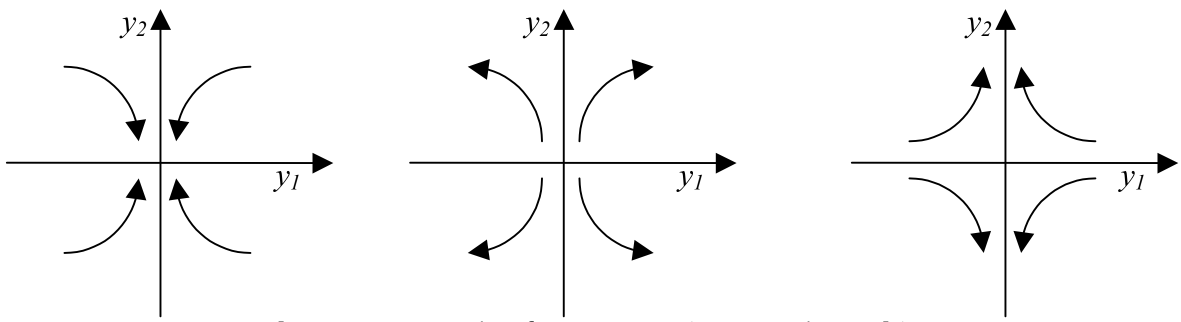
7.1 Constant coefficient linear differential equations systems

Every non-homogenous system $\dot{x} = Ax + b$ can be transformed to homogenous system $y' = Ay$ by the means of a transformation *($x \rightarrow y$, $y = x + A^{-1}b$)*

Generally the condition for convergence is $Re(\lambda) < 1$ for all λ, and complementarily if $Re(\lambda) > 1$ for some λ, the system is unstable as the corresponding variable tends to diverge. The last possible case, i.e. $Re(\lambda) = 0$ and $Im(\lambda) \neq 0$ for some λ, implies neutral stability in the corresponding two variables[37], which oscillate around the origin.

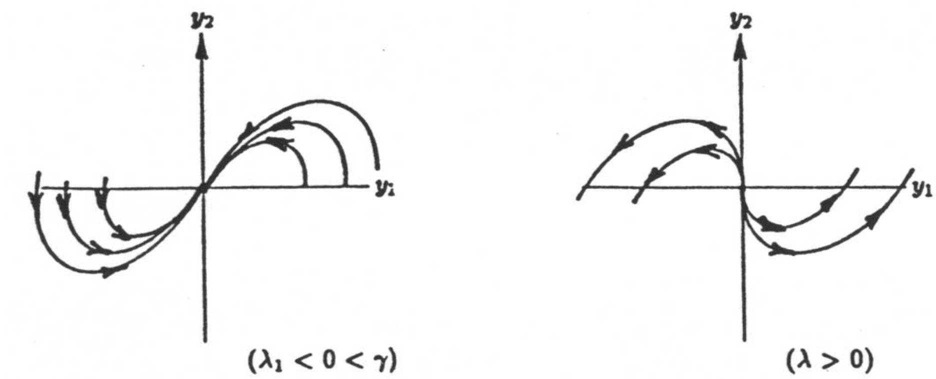
The two-dimensional case are summarized on the graphs below.

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ real distinct eigenvalues



Phase space portraits of an attractor (an attracting node), a repeller (a repelling node) and a saddle point.

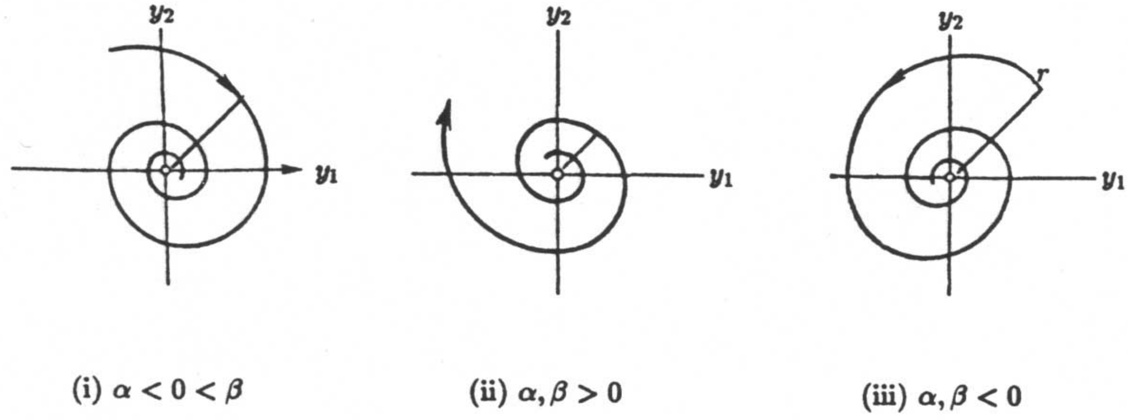
$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ repeated eigenvalues



Phase space portraits of improper attracting and repelling nodes. Picture [Tu], p.138

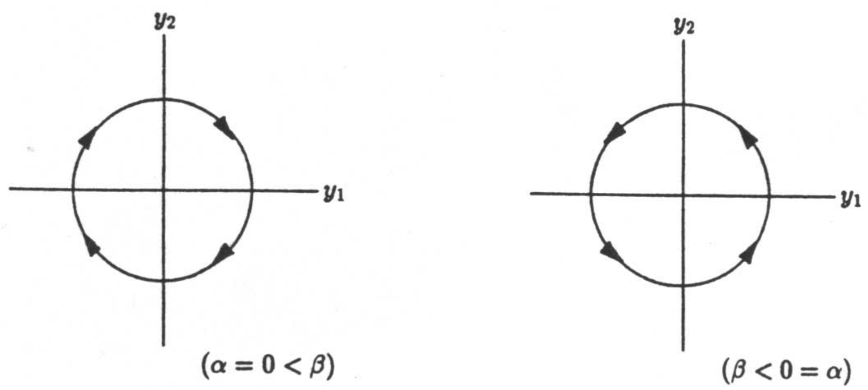
[37] As eigenvalues are the roots of polynomials, they come in complex cojugate pairs $\lambda_{1,2} = \alpha \pm i\beta$, i.e. $Re(\lambda_1) = Re(\lambda_2)$.

$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ complex eigenvalues : non-zero imaginary part ($Im(\lambda) = \beta$) implies rotation around the origin, the sign of the real part ($Re(\lambda) = \alpha$) determines convergence/divergence again, i.e. converging or diverging vortex. Saddle points are impossible (see footnote 32).



Spiraling attractors and a repeller (in the middle) for complex eigenvalues. Picture [Tu], p.138

$\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ In the special case of purely imaginary eigenvalues, the resulting phase-space portrait is a neutral center.



Neutral centers. Picture [Tu], p.138

The following table classifies the fixed points of a two-dimensional system according to the sigh of its eigenvalues' real parts, i.e. it disregards possible oscillations/rotations caused by non-zero imaginary parts.

Re(λ_1)	Re(λ_2)	Critical point
+	+	Repeller
+	-	Saddle point
-	-	Attractor
+	0	1D – Repeller
-	0	1D – Attractor
0	0	Stable point

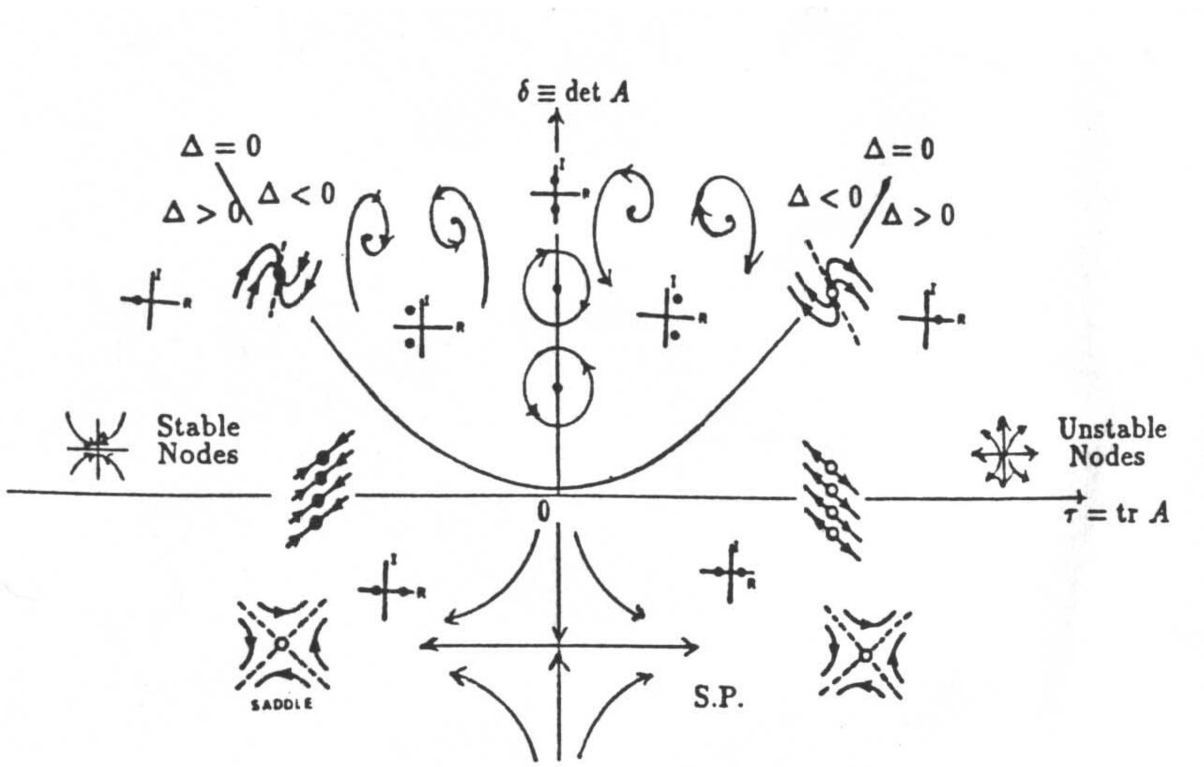
7.2 The trace-determinant-discriminant (τ-δ-Δ) method

Better understanding of linear 2-D systems is achieved, if the following definitions are accepted.

trace: $\tau = tr(A) = a_{11} + a_{22}$, *determinant*: $\delta = det(A)$, *discriminant*: $\Delta = \tau^2 - 4\delta$

which imply $\lambda_i = 1/2(\tau \pm \Delta^{1/2})$.

The solution types for the case n = 2 are summarized by the following table and graphs in the two-dimensional trace-determinant phase space. It can be easily observed that solution type of the two-dimensional system changes when τ, Δ or δ change sign.



Eigenvalue structure and corresponding solution type in the trace-determinant phase space
Picture [Tu], p.138

7.3 Reduction of n-th order ODEs to first order n-variable system[38]

An n-th order differential equation $x^{(n)}(t) + a_1 x^{(n-1)}(t) + \cdots + a_{n-1}x'(t) + a_n x(t) = 0$ can be reduced to a system $\bar{y}' = A\bar{y}$ by the following re-definition

$y_1 \equiv x$

$y_2 \equiv x' \equiv y_1'$

$y_3 \equiv x'' \equiv y_2'$ etc.

which is simply expressed by the *companion matrix* A.

$$A \equiv \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_2 & -a_1 \end{pmatrix}$$

[38] see [Tu], p. 98

7.4 Linearization and its failures

Non-linear systems $\dot{x} = f(x)$ are not globally stable in general as opposed to the linear systems, however they may under certain circumstances by locally approximated by linear systems.

Definitions:

Given the system $\dot{x} = f(x)$ the point x^* where $f(x^*) = 0$ is called a ***critical*** or ***fixed point***, as opposed to ***regular points*** where $f(x) \neq 0$.

Fixed point is called ***simple***, if its linearization A has no zero eigenvalues, i.e. $\det A \neq 0$.

Simple point with linearization A, whose eigenvalues have no zero real parts (i.e. $Re(\lambda_i) \neq 0$ for all indices i) is called ***hyperbolic***.

7.4.1 Hartman & Grobman Linearization Theorem[39]

Let the non-linear dynamic system $\vec{x}' = \vec{f}(\vec{x}, t) : U \subset R^n \rightarrow R^n$ have a simple *hyperbolic fixed point* x^*. Then in the neighborhood U of x^*, the phase portraits of the non-linear system and its linearization $\vec{x}' = A\vec{x}$ are equivalent, i.e. the systems are qualitatively equivalent.

Consequence: near a simple hyperbolic point, the dynamical system is structurally stable.

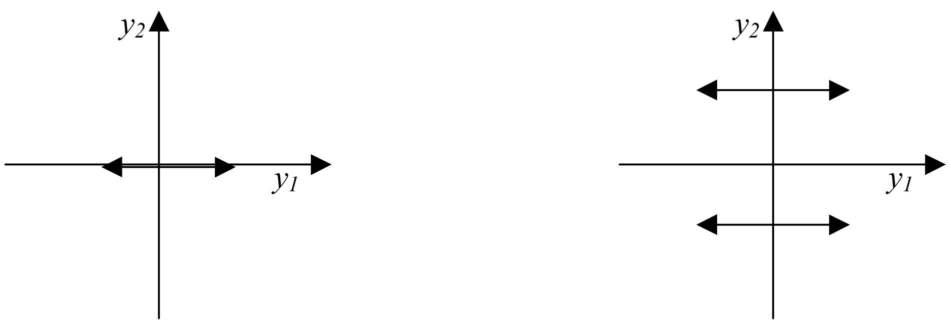
(near in the sense of mathematical analysis)

7.4.2 Bifurcation Theory Theorem[40]

Let $f(x,p): R^2 \to R$ be a smooth mapping and $f(x^*,p) = 0$ for all p, $df/dx(x^*,p^*) = 0$ and $d^2f/dxdp(x^*,p^*) \neq 0$, then the point (x^*,p^*) is a bifurication point and the set of solutions of $f(x,p)$ consists of two crossing curves.

In other words, if the real part of some eigenvalue $Re(\lambda_1(p))$ of the linearized system as a function of parameter p crosses the imaginary axis for $p = p^*$ at non-zero speed, there is an exchange of stability and a bifurcation is said to take place.

In the one-dimensional case, the crossing of the imaginary axis has to take place on the real line, for higher dimension also crossings off the real line are allowed.



On the real line (on the left) and off the real line (on the right) eigenvalue crossings.

[39] see [Tu], p. 135-6

[40] see [Tu], p.199 and p. 195-204

8 Appendix 2 - SEF program

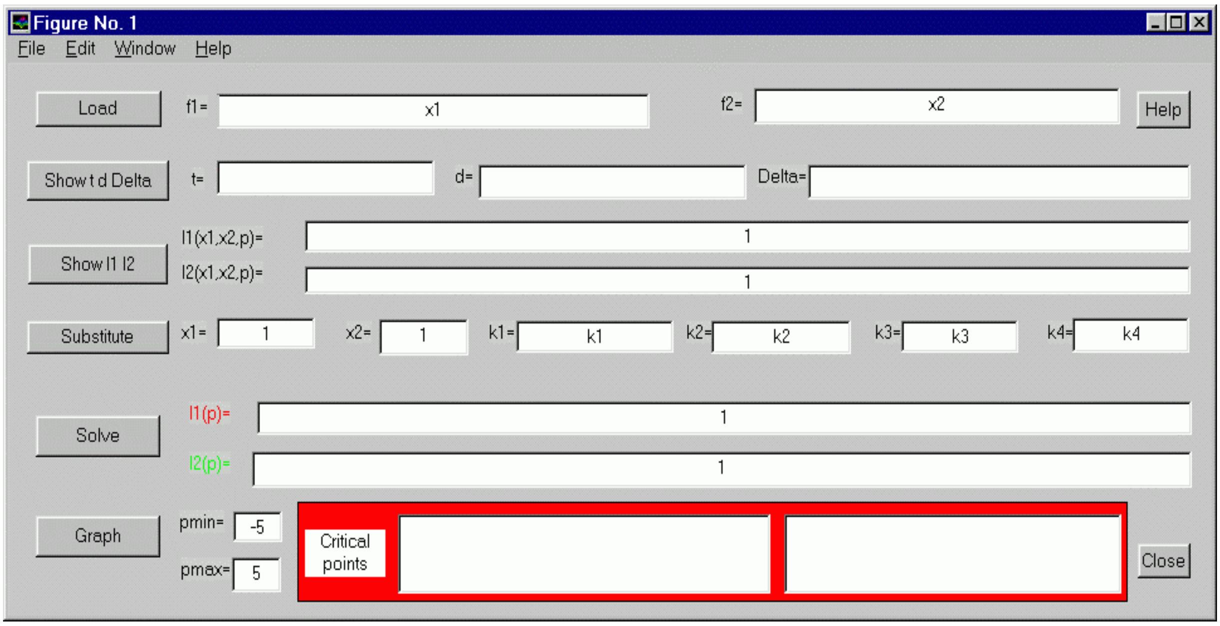
8.1 MATLAB and SEF

MATLAB is a powerful tool for handling mathematical objects such as vectors, matrixes and even differential equations. These objects do not have to be real data but may be general enough to be defined symbolically in terms of algebra, which is particularly useful for studying economical models as these are often specified in terms of more or less general functional relations.

The SEF (Simple Eigenvalue Finder) program is a simple script to be executed in the MATLAB environment with Symbolic Math Toolbox installed. SEF program can be used to analyze any one- or two-dimensional mathematical model. Should higher dimension models be analyzed, the conversion of the program is by no means difficult. Help is available upon the pressing of the 'Help' button, which shows the initial comments of the script (lines 1-15).

One-dimensional systems can easily be investigated if the second function is set to x_2, as in this case the first eigenvalue will be the eigenvalue of the 1-D system and the second eigenvalue will always be one[41].

8.2 SEF window image



8.3 SEF program source

```
function sef(action)
%SEF
%Version 2.1 (now the main window source is included in the script)
%Copyright Jiri Sumbera
%Simple Eigenvalue Finder. Calculates symbolically the eigenvalues  of the
%derivative of the vector function f=(f1,f2).
%!!! All editable fields can be edited !!! Just play around.
%Load - loads up the vector function f from file mydata.m
%!!! These buttons will work only if corresponding input fields are filled in.
%Show - calculates the eigenvalues of the derivative matrix as functions of
```

[41] since a_{21} and a_{12} are zero, a_{22} is one the characteristic polynomial will be $(\lambda - df/dx).(\lambda - 1)$.

```
%coordinates and parameter t
%Substitute - substitutes the specified values of coordinates into the eigenvalues
%Solve - solves the resulting equation Re(lambda(t))=0.
%Graph - plots eigenvalues as complex functions of t
%Conversion to vector function f=(f1,f2,f3) will be made soon.
%the variables x1 and x2, constants k1-k4, parameter p, trace t,
% determinant d and discriminant Delta are created.
syms x1 x2 k1 k2 k3 k4 p t d Delta real;

%starts the program if no input arguments are given
if nargin == 0
%%%%%%%%%%%%%%%%%%%%%%%%%%%  Initialization section.
% the main window seffigure and its fields are initialized and drawn.

seffigure = figure('Color',[0.8 0.8 0.8], 'Colormap','default',
      'Position',[14 324 957 397], 'Tag','Fig1');
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef load',
      'Position',[13.6552 217.862 62.069 16.1379], 'String','Load',
      'Tag','Pushbutton1');
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef close',
      'Position',[557.69 16.7586 26.069 15.5172], 'String','Close',
      'Tag','Pushbutton2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[103.655 217.241 212.897 15.5172], 'String','x1',
      'Style','edit', 'Tag','f1', 'UserData','x1*x2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[368.069 218.483 181.241 16.1379], 'String','x2',
      'Style','edit', 'Tag','f2');
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef
      showl1l2', 'Position',[9.93106 147.724 69.5172 18], 'String','Show l1
      l2', 'Tag','Pushbutton3');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[146.483 162 436.966 13.6552], 'String','1',
      'Style','edit', 'Tag','lambda1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[146.483 143.379 436.966 12.4138], 'String','1',
      'Style','edit', 'Tag','lambda2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[83.7931 217.862 21.1034 14.2759],
      'String','f1=', 'Style','text', 'Tag','f1label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[347.586 218.483 20.4828 14.2759],
      'String','f2=', 'Style','text', 'Tag','f2label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[83.1724 159.517 46.5517 14.2759],
      'String','l1(x1,x2,p)=', 'Style','text', 'Tag','lambda1label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[83.7931 142.759 45.3103 14.8966],
      'String','l2(x1,x2,p)=', 'Style','text', 'Tag','lambda2label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[83.7931 116.069 18 14.8966], 'String','x1=',
      'Style','text', 'Tag','x1label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[165.103 116.069 18 15.5172], 'String','x2=',
      'Style','text', 'Tag','x2label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[103.655 118.552 48.4138 13.6552], 'String','1',
      'Style','edit', 'Tag','x1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[183.724 115.759 44.069 16.1379], 'String','1',
      'Style','edit', 'Tag','x2');
```

```
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef graph',
      'Position',[13.9655  25.4483  61.4483  18.6207],  'String','Graph',
      'Tag','Pushbutton4');
b  =  uicontrol('Parent',seffigure,  'Units','points',  'Callback','hthelp
      sef', 'Position',[557.069 217.241 26.6897 16.7586], 'String','Help',
      'Tag','Pushbutton5');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],  'Position',[122.897  80.069  461.172  15.5172],  'String','1',
      'Style','edit', 'Tag','l1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],  'Position',[121.034  57.1034  463.655  16.1379],  'String','1',
      'Style','edit', 'Tag','l2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8  0.8],  'ForegroundColor',[1  0  0],  'Position',[83.7931  79.4483
      32.2759 16.1379], 'String','l1(p)=', 'Style','text', 'Tag','l1text');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8  0.8],  'ForegroundColor',[0  1  0],  'Position',[83.7931  57.1034
      32.2759 16.1379], 'String','l2(p)=', 'Style','text', 'Tag','l2text');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8   0.8],   'Position',[83.7931   29.7931   24.8276   17.3793],
      'String','pmin=', 'Style','text', 'Tag','tminlabel');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8   0.8],   'Position',[83.7931   9.93103   29.1724   16.1379],
      'String','pmax=', 'Style','text', 'Tag','tmaxlabel');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],  'Position',[111.724  32.2759  24.2069  13.6552],  'String','-5',
      'Style','edit', 'Tag','pmin');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],  'Position',[111.103  10.5517  24.2069  15.5172],  'String','5',
      'Style','edit', 'Tag','pmax');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],   'Max',2,   'Position',[383.586   9.93103   166.345   35.3793],
      'Style','edit', 'Tag','solution2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],   'Max',2,   'Position',[192.414   9.93103   183.724   35.3793],
      'Style','edit', 'Tag','solution1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[147.103 17.069 39.7241 21.1034], 'String','Critical
      points', 'Style','text', 'Tag','solutionslabel1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 0
      0],  'Position',[143.379  5.58621  409.655  44.6897],  'Style','frame',
      'Tag','Frame1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8  0.8],  'Position',[234.621  116.379  18  14.8966],  'String','k1=',
      'Style','text', 'Tag','k1label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8  0.8],  'Position',[332.069  116.069  18  15.5172],  'String','k2=',
      'Style','text', 'Tag','k2label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],  'Position',[250.759  116.69  77.5862  14.2759],  'String','k1',
      'Style','edit', 'Tag','k1');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1],  'Position',[347.586  116.379  68.8966  14.8966],  'String','k2',
      'Style','edit', 'Tag','k2');
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef solve',
      'Position',[13.9655  70.7586  61.4483  18.6207],  'String','Solve',
      'Tag','Pushbutton4');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8  0.8],  'Position',[425.793  116.069  18  15.5172],  'String','k3=',
      'Style','text', 'Tag','k3label');
```

```
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[510.207 116.069 18 15.5172], 'String','k4=',
      'Style','text', 'Tag','k4label');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[440.69 116.69 57.7241 14.2759], 'String','k3',
      'Style','edit', 'Tag','k3');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[525.724 116.69 57.7241 15.5172], 'String','k4',
      'Style','edit', 'Tag','k4');
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef subst',
      'Position',[9.62071 116.069 70.1379 15.5172], 'String','Substitute',
      'Tag','Pushbutton6');
b = uicontrol('Parent',seffigure, 'Units','points', 'Callback','sef
      showtdD', 'Position',[9.62071 184.345 70.1379 18], 'String','Show t d
      Delta', 'Tag','Pushbutton3');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[83.7931 185.586 21.1034 14.2759],
      'String','t=', 'Style','text', 'Tag','tlabel');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[215.379 186.207 21.1034 14.2759],
      'String','d=', 'Style','text', 'Tag','dlabel');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[0.8
      0.8 0.8], 'Position',[368.69 186.207 26.6897 14.2759],
      'String','Delta=', 'Style','text', 'Tag','Deltalabel');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[103.655 187.448 106.759 15.5172], 'Style','edit',
      'Tag','t', 'UserData','x1*x2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[232.138 185.586 132.207 15.5172], 'Style','edit',
      'Tag','d', 'UserData','x1*x2');
b = uicontrol('Parent',seffigure, 'Units','points', 'BackgroundColor',[1 1
      1], 'Position',[394.759 185.586 188.69 15.5172], 'Style','edit',
      'Tag','Delta', 'UserData','x1*x2');
%%%%%%%%%%%%%%%%%%%%%%%%%%  end of Initialization section
else
switch(action)
%the main switch function splitting the different tasks of the program
      case 'load',
      %loads data from file mydata.m
      sefdata;
      handlef1 = findobj(gcbf,'Tag','f1');
      set(handlef1,'String',f1);
      handlef2 = findobj(gcbf,'Tag','f2');
      set(handlef2,'String',f2);
      case 'showtdD'
      %calculates trace t, determinant d and diskriminant Delta
      %of the linearization matrix A
         handlef1 = findobj(gcbf,'Tag','f1');
         f1=sym(get(handlef1,'String'));
         handlef2 = findobj(gcbf,'Tag','f2');
          f2=sym(get(handlef2,'String'));
         df11=diff(f1,x1);    df12=diff(f1,x2);
         df21=diff(f2,x1);    df22=diff(f2,x2);
         A=[df11 df12; df21 df22];
         lambda=eig(A);
         handlet = findobj(gcbf,'Tag','t');
         t=A(1,1)+A(2,2);
         set(handlet,'String',char(t));
         handled = findobj(gcbf,'Tag','d');
         d=det(A);
         set(handled,'String',char(d));
```

```
      handleDelta = findobj(gcbf,'Tag','Delta');
      Delta=t^2-4*d;
      set(handleDelta,'String',char(Delta));
   case 'showl1l2'
   %creates the square derivative matrix of f=(f1,f2)
   %and calculates its eigenvalues which are shown in fields lambda 1,2
      handlef1 = findobj(gcbf,'Tag','f1');
      f1=sym(get(handlef1,'String'));
      handlef2 = findobj(gcbf,'Tag','f2');
       f2=sym(get(handlef2,'String'));
      df11=diff(f1,x1);          df12=diff(f1,x2);
      df21=diff(f2,x1);          df22=diff(f2,x2);
      A=[df11 df12; df21 df22];
     lambda=eig(A);
      handlelambda1 = findobj(gcbf,'Tag','lambda1');
      set(handlelambda1,'String',char(lambda(1)));
      handlelambda2 = findobj(gcbf,'Tag','lambda2');
      set(handlelambda2,'String',char(lambda(2)));
   case 'subst',
   %substitutes the specified values of x1, x2 and the constants
   %k1 and k2 as specified by the input fields into the eigenvalues
      handlelambda1 = findobj(gcbf,'Tag','lambda1');
      handlelambda2 = findobj(gcbf,'Tag','lambda2');
      lambda1 = sym(get(handlelambda1,'String'));
      lambda2 = sym(get(handlelambda2,'String'));
      handlex1 = findobj(gcbf,'Tag','x1');
      x1 = sym(get(handlex1,'String'));
      handlex2 = findobj(gcbf,'Tag','x2');
      x2 = sym(get(handlex2,'String'));
      lambda1 = subs(lambda1);   lambda2 = subs(lambda2);
      handlek1 = findobj(gcbf,'Tag','k1');
      k1 = sym(get(handlek1,'String'));
      handlek2 = findobj(gcbf,'Tag','k2');
      k2 = sym(get(handlek2,'String'));
      handlek3 = findobj(gcbf,'Tag','k3');
      k3 = sym(get(handlek3,'String'));
      handlek4 = findobj(gcbf,'Tag','k4');
      k4 = sym(get(handlek4,'String'));
      lambda1 = subs(lambda1);   lambda2 = subs(lambda2);
      handlel1 = findobj(gcbf,'Tag','l1');
      handlel2 = findobj(gcbf,'Tag','l2');
      set(handlel1,'String',char(lambda1));
      set(handlel2,'String',char(lambda2));
    case 'solve', %finds the solution to the equation Re(lambda(t))= 0
      handlel1 = findobj(gcbf,'Tag','l1');
      handlel2 = findobj(gcbf,'Tag','l2');
      lambda1=sym(get(handlel1,'String'));
      lambda2=sym(get(handlel2,'string'));
      s1=solve(real(lambda1),p);
      s2=solve(real(lambda2),p);
      handlesol1=findobj(gcbf,'Tag','solution1');
      handlesol2=findobj(gcbf,'Tag','solution2');
      set(handlesol1,'String',char(s1));
      set(handlesol2,'String',char(s2));
   case 'graph', %gets the range values of t and plots a 3D-graph
                 %with lambda in the complex plane and p on the z-axis
      scrsz = get(0,'ScreenSize');
      figf = figure('Position',[40 40  scrsz(3)/1.5 scrsz(4)/1.5],...
           'menu','none','tag','figf');
        figure(figf)
        uicontrol('pos',[25 60 32 20],'string','Up', ...
```

```
            'CallBack','sef up');
            uicontrol('pos',[10 35 30 20],'string','<=', ...
            'CallBack','sef right');
            uicontrol('pos',[40 35 30 20],'string','=>', ...
            'CallBack','sef left');
            uicontrol('pos',[20 10 40 20],'string','Down', ...
            'CallBack','sef down');
         handlepmin = findobj(gcbf, 'Tag','pmin');
         handlepmax = findobj(gcbf, 'Tag','pmax');
         p_min = str2num(get(handlepmin,'String'));
         p_max = str2num(get(handlepmax,'String'));
         handlel1 = findobj(gcbf,'Tag','l1');
         handlel2 = findobj(gcbf,'Tag','l2');
         lambda1=sym(get(handlel1,'String'));
         lambda2=sym(get(handlel2,'string'));
         p = p_min:(p_max-p_min)/40:p_max;
         y1 = double(subs(lambda1));
         if prod(size(y1))==1
            y1=y1*ones(1,41);
            end
         y2 = double(subs(lambda2));
         if prod(size(y2))==1
            y2=y2*ones(1,41);
            end
          handleline1=plot3(real(y1),imag(y1),p,'r-');
          set(handleline1,'LineWidth',2) ;
         hold on;
        handleline2= plot3(real(y2),imag(y2),p,'g-');
         set(handleline2,'LineWidth',2);
        grid on;
         set(gca,'Box','On');
         set(gca,'Xlabel',text('String','Re(lambda)'));
         set(gca,'Ylabel',text('String','Im(lambda)'));
         set(gca,'Zlabel',text('String','p'));
         sef cut; hold off;
         view([-36 20]);
       case 'cut',
          hold on;
          Yr = get(gca,'YLim');
         Zr = get(gca,'ZLim');
         Y =[Yr' Yr']; Z= [Zr ; Zr];
         surf(zeros(2), Y, Z);
         colormap(jet);
         hold off;
      %graph view rotations
      case 'left',
         [V(1), V(2)]=view;
         view([V(1)-18 V(2)]);
      case 'right',
         [V(1), V(2)]=view;
         view([V(1)+18 V(2)]);
      case 'up',
         [V(1), V(2)]=view;
         view([V(1) V(2)+10]);
      case 'down',
         [V(1), V(2)]=view;
         view([V(1) V(2)-10]);
      case 'close', %closes the SEF program
         close(gcbf)
  end
end
```